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On Symmetric n-Multipliers of Semiprime Rings

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Abstract: Many scientific studies indicate the great relationship between the multiplier (centralizer) and the structure of associative rings. In particular, the concept of multiplier accurately overlaps with several algebraic terms such as the derivation and their generalizations that have wide applications. The main purpose of this paper is to generalize the term of multiplier to n-multiplier and prove many results by introducing specific concepts concerning it and study their properties on semiprime ring.

Keywords: semiprime ring, multiplier, Jordan multiplier, n-multiplier, left n-multiplier, Jordan n-multiplier.

I. INTRODUCTION

Throughout this paper, $\mathcal{R}$ will represent an associative ring with center $Z(\mathcal{R})$. A ring $\mathcal{R}$ is said to be n-torsion free when $na = 0$with $a \in \mathcal{R}$, then $a = 0$ where n is a nonzero integer [1]. As usual, the commutator $uy - yu$will be denoted by $[u, y]$. We shall use the basic commutator identities $[uy, w]$=$[u, w]y + u[y, w]$ and $[u, yw]$=$[u, y]w + y[u, w]$. A ring $\mathcal{R}$ is said to be prime if $a\mathcal{R}b = 0$ implies that either $a = 0$ orb=0 for all a, b∈$\mathcal{R}$. A ring $\mathcal{R}$ is said to be semiprime if $a\mathcal{R}a = 0$ implies thata=0 for all $a \in \mathcal{R}$ [2].An additive subgroup U of $\mathcal{R}$ is called Lie ideal if whenever $u \in U$, $r \in \mathcal{R}$ then $[U, r]$∈U [10]. In [3], Zalar introduced the term of centralizer (multiplier) and the author proved many results concerning multiplier. An additive mapping $\mathcal{M}$ is called left (resp. right) multiplier if $\mathcal{M}(uy) = \mathcal{M}(u)y$ (resp. $\mathcal{M}(uy) = u\mathcal{M}(y)$) holds for all $u, y \in \mathcal{R}$, and $\mathcal{M}$ is called a multiplier if it is a left and right multiplier [4]. Further, an additive mapping $\mathcal{M}: \mathcal{R} \to \mathcal{R}$ is called a left.(resp. right) Jordan multiplier in case $\mathcal{M}(u^2) = \mathcal{M}(u)u$ (resp. $\mathcal{M}(u^2) = u\,\mathcal{M}(u)$) holds for $u \in \mathcal{R}$ [5]. During last few years, many authors discussed severalproperties of multiplier and some generalizations related to the concept of multiplier are made ([6], [7] and [8]). Clearly, every left (resp. right) multiplier is a Jordan left (resp. right) multiplier. Zalar proved that every left (resp. right) Jordan multiplier on a 2-torsion free semiprime ring is a left (resp. right) multiplier [3]. Shakir and Claus in [10] have proved that if an additive mapping $\mathcal{M}$of , a 2-torsion free semiprime ring $\mathcal{R}$ satisfying $\mathcal{M}(\text{uyu}) = \mathcal{M}(u)\text{yu}$ (resp. $\mathcal{M}(\text{uyu}) = uy\mathcal{M}(\text{u})$)for all $u, y \in \mathcal{R}$, then $\mathcal{M}$ is a left (resp. right) multiplieron $\mathcal{R}$. In 2001, Vukman proved that if $\mathcal{R}$ is a 2-torsion free semiprime ring such that $\mathcal{M}(uyu) = u\mathcal{M}(\text{y})u$ for all $u, y \in R$, then $\mathcal{M}$ is multiplier [5]. Further, Vukman and Kosi-Ulbl in [8] proved that if $\mathcal{M}: \mathcal{R} \to \mathcal{R}$ is an additive mapping such that $2\mathcal{M}(uyu) = \mathcal{M}(u)yu + uy\mathcal{M}(u)$ for all $u, y \in R$, then $\mathcal{M}$ is multiplier. Also, there are authors who have shown results for multiplier withLie ideal as in ([12], [13]).This paper is organized as follows. Section 2 is devoted to recalling somemathematical preliminaries and fundamental facts of left (right) n-multiplier and n-multiplier. Section 3 presents a generalization of several results that appear in ([3], [5], [10]).

II. PRELIMINARIES

Firstly, we begin with some basic facts which are important to get our main results concerning with a symmetric n-multiplier of prime and semiprime ring:

Lemma 2.1[4]: Let $\mathcal{R}$ be a semiprime ring. Suppose that the relation $axb + bxc = 0$ holds for all $x \in \mathcal{R}$ and some, $b, c \in \mathcal{R}$. In this case $axb = bxa = 0$ is satisfied for all $x \in \mathcal{R}$.

Lemma 2.2 [3]: Let $\mathcal{R}$ be a semiprime ring if a, b∈ $\mathcal{R}$ are such that aub=0 for all u∈ $\mathcal{R}$, then ab=ba=0.

Lemma 2.3 [3]: Let $\mathcal{R}$ be a semiprime ring and "a" be an element of $\mathcal{R}$. If $a[u, y]$ =0 for each $u, y \in \mathcal{R}$, then there exists an ideal U of $\mathcal{R}$ such that $a \in U \in Z$ holds.

Lemma 2.4 [9]: Let$\mathcal{R}$ be a prime ring with char $(\mathcal{R}) \neq 2$ and U a non-central Lie ideal of $\mathcal{R}$. If for some a, b∈ $\mathcal{R}$, $aUb = 0$, then $a = 0$ or $b = 0$.

Lemma 2.5 [9]: Let$\mathcal{R}$ be a prime ring with char $(\mathcal{R}) \neq 2$ and U a noncentral Lie ideal of $\mathcal{R}$. Then $C_{\mathcal{R}}(U)$=$Z(\mathcal{R})$.

Definition 2.6 [11]: Amapping $\mathcal{M}: \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R} \to \mathcal{R}$ is called symmetric if the equation $\mathcal{M}(u^2, u_2, \ldots, u_n)$= $\mathcal{M}(u_{\pi(1)}, u_{\pi(2)}, \ldots, u_{\pi(n)})$ holds for allu_i∈$\mathcal{R}$ and for every permutation {π(1),π(2),…,π(n)}.

Now, we introduce the concepts of left (right) n-multiplier and n-multiplier.

Definition 2.7: An n-additive mapping $\mathcal{M}:\mathcal{R}\times\mathcal{R}\times\ldots\times\mathcal{R}\to\mathcal{R}$ is said to be left n-multiplier if the following equations hold for all y , $u_1, u_2, \ldots, u_n \in \mathcal{R}$,

$\mathcal{M}_1(u_1 y, u_2, \ldots, u_n) = \mathcal{M}_1(u_1, u_2, \ldots, u_n)y$;

$\mathcal{M}_2(u_1, u_2 y, \ldots, u_n) = \mathcal{M}_2(u_1, u_2, \ldots, u_n)y$;

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$\mathcal{M}_n(u_1, u_2, \ldots, u_n y) = \mathcal{M}_n(u_1, u_2, \ldots, u_n)y$.

$\mathcal{M}$ is said to be a symmetric left n-multiplier if all the above equations are equivalent to all other. That is, $\mathcal{M}(u_1 y, u_2, \ldots, u_n) = \mathcal{M}(u_1, u_2, \ldots, u_n)y$ for all $y, u_1, u_2, \ldots, u_n \in \mathcal{R}$.

Also, $\mathcal{M}$ is said to be a symmetric right n-multiplier if all the above Equations are equivalent to all other. That is, $\mathcal{M}(u_1 y, u_2, \ldots, u_n) = y\mathcal{M}(u_1, u_2, \ldots, u_n)$ for all $y, u_1, u_2, \ldots, u_n \in \mathcal{R}$.

Definition 2.8: An n-additive symmetric mapping $\mathcal{M}:\mathcal{R}\times\mathcal{R}\times\ldots\times\mathcal{R}\to\mathcal{R}$ is said to be a symmetric n-multiplier if $\mathcal{M}$ is symmetric left and right n-multiplier.

$\mathcal{M}$ is said to be a Jordan n-multiplier if for all $y, u_1, u_2, \ldots, u_n \in \mathcal{R}$, $\mathcal{M}(u_1{}^2, u_2, \ldots, u_n) = \mathcal{M}(u_1, u_2, \ldots, u_n)u_1$

The following example explains the notion of symmetric n-multiplier of a ring $\mathcal{R}$.

Example 2.9: Consider $\mathcal{R} = \left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}\right\}$ be a ring and $\mathbb{Z}$ is a ring of integer numbers. Let $\mathcal{M}:\mathcal{R}\times\mathcal{R}\times\ldots\times\mathcal{R}\to\mathcal{R}$ be a mapping defined by $\mathcal{M}\left(\begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & a_n \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & a_1 a_2 \ldots a_n \\ 0 & 0 \end{pmatrix}$, for all $\begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & a_n \\ 0 & 0 \end{pmatrix} \in \mathcal{R}$. Then $\mathcal{M}$ is a symmetric n-multiplier.

III. MAIN RESULTS

To prove our main results we start with following lemma which is generalization of [3, Lemma 1.2]

Lemma 3.1: Let $\mathcal{R}$ be a semiprime ring and $\mathcal{M}, \mathcal{N}:\mathcal{R}\times\mathcal{R}\times\ldots\times\mathcal{R}\to\mathcal{R}$ be a symmetric n-additive mapping, if $\mathcal{M}((u_1, u_2, \ldots, u_n)\mathcal{WN}(u_1, u_2, \ldots, u_n) = 0$ for all $u_1, u_2, \ldots, u_n \in \mathcal{R}$. Then

$\mathcal{M}((u_1, u_2, \ldots, u_n)\mathcal{WN}(\gamma_1, \gamma_2, \ldots, \gamma_n) = 0$ for all $\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathcal{R}$.

Proof:

Since $\mathcal{M}((u_1, u_2 \ldots, u_n)\mathcal{WN}(u_1, u_2 \ldots, u_n) = 0$

Replacing u_1 by $u_1 + y_1$ in the last relation to get the following

$0 = \mathcal{M}((u_1, u_2, \ldots, u_n)\mathcal{WN}(u_1, u_2, \ldots, u_n) + \mathcal{M}(u_1, u_2, \ldots, u_n)\mathcal{WN}(\gamma_1, u_2, \ldots, u_n) +$
$\mathcal{M}(\gamma_1, u_2, \ldots, u_n)\mathcal{WN}(u_1, u_2, \ldots, u_n) + \mathcal{M}(\gamma_1, u_2, \ldots, u_n)\mathcal{WN}(\gamma_1, u_2, \ldots, u_n$.

Hence, for all $\gamma_1, u_1, u_2, \ldots, u_n \in \mathcal{R}$;

$\mathcal{M}(u_1, u_2, \ldots, u_n)\mathcal{WN}(\gamma_1, u_2, \ldots, u_n) = \mathcal{M}(\gamma_1, u_2, \ldots, u_n)\mathcal{WN}(u_1, u_2, \ldots, u_n)$

Since $\mathcal{M}$ and $\mathcal{N}$ are n-additive then multiply by $\mathcal{AM}(u_1, u_2, \ldots, u_n)\mathcal{WN}(\gamma_1, u_2, \ldots, u_n)$ the both sides of the last relation from right $\mathcal{M}(u_1, u_2, \ldots, u_n)\mathcal{WN}(\gamma_1, u_2, \ldots, u_n)\mathcal{AM}((u_1, u_2, \ldots, u_n)\mathcal{WN}(\gamma_1, u_2 \ldots, u_n) = -\mathcal{M}(\gamma_1, u_2 \ldots, u_n)\mathcal{WN}(u_1, u_2 \ldots, u_n)\mathcal{AM}(u_1, u_2 \ldots, u_n)\mathcal{WN}(\gamma_1, u_2 \ldots, u_n)$. This implies that $\mathcal{M}(u_1, u_2 \ldots, u_n)\mathcal{WN}(\gamma_1, u_2 \ldots, u_n)\mathcal{AM}(u_1, u_2 \ldots, u_n)\mathcal{WN}(\gamma_1, u_2 \ldots, u_n) = 0$. By semiprimenessof $\mathcal{R}$, then $\mathcal{M}(u_1, u_2 \ldots, u_n)\mathcal{WN}(\gamma_1, u_2 \ldots, u_n) = 0$

In the same way one can replace $u_i = u_i + \gamma_i$ for all $\gamma_i \in \mathcal{R}$ where $i = 2, \ldots, n$ to get, $\mathcal{M}(u_1, u_2 \ldots, u_n)\mathcal{WN}(\gamma_1, \gamma_2, \ldots, \gamma_n) = 0$.

Proposition 3.2: Let $\mathcal{R}$ be a semiprime ring of char$\neq 2$ and $\mathcal{M}:\mathcal{R}\times\mathcal{R}\times\ldots\times\mathcal{R}\to\mathcal{R}$ be an additive mapping satisfying $\mathcal{M}(u^2, u_2, \ldots, u_n) = \mathcal{M}(u, u_2, \ldots, u_n)u$ (resp. $\mathcal{M}(u^2, u_2, \ldots, u_n) = u\mathcal{M}(u, u_2, \ldots, u_n)$) for all $u, u_2, \ldots, u_n \in \mathcal{R}$. Then $\mathcal{M}$ is a left n-multiplier.

Proof:

Since $\mathcal{M}(u^2, u_2, \ldots, u_n) = \mathcal{M}(u, u_2, \ldots, u_n)u$. … (1)

Substituting $u = u + y$ in Equation (1) to get $\mathcal{M}(uy + yu, u_2, \ldots, u_n) = \mathcal{M}(u, u_2, \ldots, u_n)y + \mathcal{M}(y, u_2, \ldots, u_n)u$. … (2)

Put $y = uy + yu$ in Equation (2), and use it to get

$\mathcal{M}(u(uy + yu) + (yu + yu)u, u_2, \ldots, u_n)$
$= \mathcal{M}(u, u_2, \ldots, u_n)uy + \mathcal{M}(u, u_2, \ldots, u_n)yu + \mathcal{M}(u, u_2, \ldots, u_n)yu + \mathcal{M}(y, u_2, \ldots, u_n)u^2$. … (3)

On other hand,

$\mathcal{M}(u(uy + yu) + (yu + yu)u, u_2, \ldots, u_n) = \mathcal{M}(u^2, u_2, \ldots, u_n)y + \mathcal{M}(y, u_2, \ldots, u_n)u^2 + 2\mathcal{M}(uyu, u_2, \ldots, u_n)$. … (4)

Comparing Equations (3) and (4), then $\mathcal{M}(uyu, u_2, \ldots, u_n) = \mathcal{M}(u, u_2, \ldots, u_n)yu$. … (5)

Linearize Equation (5) on u, $\mathcal{M}(uyw+wyu,u_2,\dots,u_n)= \mathcal{M}(u,u_2,\dots,u_n)\,yw+\mathcal{M}(w,u_2,\dots,u_n)\,yu$. … (6)
Let $H=\mathcal{M}(uywyu,u_2,\dots,u_n)+\mathcal{M}(yuwuy,u_2,\dots,u_n)$
By using Equation (5) on H we get $H= \mathcal{M}(u,u_2,\dots,u_n)ywyu+ \mathcal{M}(y,u_2,\dots,u_n)uwuy$. … (7)
Now, by using Equation (6) on H we will get $H= \mathcal{M}(uy,u_2,\dots,u_n)\,wyu+ \mathcal{M}(yu,u_2,\dots,u_n)\,wuy$. … (8)
Comparing Equations (7) and (8), we have $N(u,y)A[u,y]$ =0 where $N(u,y)$ = $\mathcal{M}(uy,u_2,\dots,u_n)$ - $\mathcal{M}(u,u_2,\dots,u_n)y$. ... (9)
By Lemma 3.1andLemma2.2, then $N(u,y)A[u',y']$=0. … (10)
Now, by Lemma 2.3**,** we get $N(u,y)A\in U,U$ is an ideal, where $U\subset Z(\mathcal{R})$. If $A=[u',y']AN(u,y)$ in Equation (10), then we have $N(u,y)[y_1,y_2]\,AN(u,y)\,[u',y']$=0.
By semiprimenessof $\mathcal{R}$, and for all $A\in\mathcal{R}$,we get $N(u,y)[u',y']=0$. Using Lemma 2.3, then $N(u,y)\in Z(\mathcal{R})$, hence $N(u,y)A\in Z(\mathcal{R})$ and $AN(u.y)\in Z(\mathcal{R})$.
In particular, $nN(u,y),N(u,y)n\in Z(\mathcal{R})$ for all $n\in\mathcal{R}$, this gives us
$u.N^2(u,y)y=N^2(u,y)y.u=yN^2(u,y).u\ =y.N^2(u,y)u$.
This gives us $4\mathcal{M}(u.N^2(u,y)y,u_2,\dots,u_n)=4\mathcal{M}(y.N^2(u,y)u,u_2,\dots,u_n)$.
Now, we will compute the both side to above Equation, $2\,\mathcal{M}(u.N^2(u,y)y+N^2(u,y)yu,u_2,\dots,u_n)=2\,\mathcal{M}(y.N^2(u,y)u+N^2(u,y)uy,u_2,\dots,u_n)$.
$2\,\mathcal{M}(u.,u_2,\dots,u_n)N^2(u,y)y\ +\mathcal{M}(N,u_2,\dots,u_n)N(u,y)yu+\mathcal{M}(y,u_2,\dots,u_n)N^2(u,y)u$
$=2\,\mathcal{M}(y,u_2,\dots,u_n)N^2(u,y)u+\mathcal{M}(\,N(u,y),u_2,\dots,u_n)N(u,y)uy+ \mathcal{M}(u,u_2,\dots,u_n)N^2(u,y)$. That is, $\mathcal{M}(u.,u_2,\dots,u_n)N^2(u,y)y\ +\mathcal{M}(N,u_2,\dots,u_n)N(u,y)yu=$
$\mathcal{M}(y,u_2,\dots,u_n)N^2(u,y)u+ \mathcal{M}(\,N,u_2,\dots,u_n)N(u,y)uy$. Since ,$N(u,y)yu=N(u,y)y.u=u.N(u,y)y=uN(u,y)y$=$N(u,y)uy$ we get, $\mathcal{M}(u.,u_2,\dots,u_n)N^2(u,y)y$= $\mathcal{M}(y,u_2,\dots,u_n)N^2(u,y)u$. Also, then again, we likewise we have, $4\mathcal{M}(uyN^2(u,y),u_2,\dots,u_n)= 4\mathcal{M}(u\,N(u,y).yN(u,y),u_2,\dots,u_n)$
$2\,\mathcal{M}(uy,u_2,\dots,u_n)\,N^2(u,y)+2\mathcal{M}(N,u_2,\dots,u_n)N(u,y)uy=$
$\mathcal{M}(u,u_2,\dots,u_n)N^2(u,y)y+\mathcal{M}(N,u_2,\dots,u_n)N(u,y)uy+ \mathcal{M}(y,u_2,\dots,u_n)N^2(u,y)u$
$+\mathcal{M}(N(u,y),u_2,\dots,u_n)N(u,y)uy$
$2\,\mathcal{M}(uy,u_2,\dots,u_n)\,N^2(u,y)$= $\mathcal{M}(u,u_2,\dots,u_n)yN^2(u,y)$+ $\mathcal{M}(y,u_2,\dots,u_n)uN^2(u,y)$
Utilizing above relation we finally arrive at, $\mathcal{M}(uy,u_2,\dots,u_n)\,N^2(u,y)$= $\mathcal{M}(u,u_2,\dots,u_n)yN^2(u,y)$.
In any case, this implies $N^3(u,y)=0$ so $N^2(u,y)\mathcal{R}N^2(u,y)=N^4(u,y)=(0)$, $N(u,y)\mathcal{R}N(u,y)=N^2(u,y)\mathcal{R}=(0)$ then $N(u,y)=0$.

Theorem 3.3: Let $\mathcal{R}$ be a 2-torsion free semiprime ring If$\mathcal{M}$**:**$\mathcal{R}\times\mathcal{R}\times\ldots\times\mathcal{R}\to\mathcal{R}$ is a n-additive mapping such that $\mathcal{M}(u\gamma u,u_2,\dots,u_n)= \mathcal{M}(u,u_2,\dots,u_n)\gamma u$ for all $\gamma,u,u_2,\dots,u_n\in\mathcal{R}$. Then $\mathcal{M}$ is a symmetric left n-multiplier on $\mathcal{R}$.
Proof:
By assumption, $\mathcal{M}(u\gamma u,u_2,\dots,u_n)$= $\mathcal{M}(u,u_2,\dots,u_n)\,\gamma u$.… (1)
Substituting $u=u+z$ in Equation (1), then
$\mathcal{M}\big((u+z)\gamma(u+z),u_2,\dots,u_n\big)=$
$\mathcal{M}(u,u_2,\dots,u_n)\gamma u+ \mathcal{M}(u,u_2,\dots,u_n)\gamma z+ \mathcal{M}(z,u_2,\dots,u_n)\gamma u+ \mathcal{M}(z,u_2,\dots,u_n)\gamma z$. … (2)
On the other hand, $\mathcal{M}\,((u+z)\gamma(u+z),u_2,\dots,u_n)$= $\mathcal{M}\,((u\gamma u+u\gamma z+z\gamma u+z\gamma z,u_2,\dots,u_n)$= $\mathcal{M}\,(u\gamma z+z\gamma u,u_2,\dots,u_n)$+ $\mathcal{M}(u,u_2,\dots,u_n)\gamma u$+ $\mathcal{M}(z,u_2,\dots,u_n)$. ...(3)
Combining Equations (2) and (3) we have
$\mathcal{M}(u\gamma z+z\gamma u,u_2,\dots,u_n)=\mathcal{M}(u,u_2,\dots,u_n)\gamma z$+ $\mathcal{M}(z,u_2,\dots,u_n)\gamma u$ for all $z,\gamma,u,u_2,\dots,u_n\in\mathcal{R}$. … (4)
When let $z=u^2$ in Equation (4) to get
$\mathcal{M}(u\gamma u^2+u^2\gamma u,u_2,\dots,u_n)$=$\mathcal{M}(u,u_2,\dots,u_n)\gamma u^2$+ $\mathcal{M}(u^2,u_2,\dots,u_n)\gamma u$. ... (5)
Now, replacing γ by uγ+γu in Equation (1) and using it will get
$\mathcal{M}(u(u\gamma+\gamma u)u,u_2,\dots,u_n)$=$\mathcal{M}(u,u_2,\dots,u_n)u\gamma u$+ $\mathcal{M}(u,u_2,\dots,u_n)\gamma u^2$. … (6)
Now, combining Equations (6) and (5) will get
$\mathcal{M}(u^2,u_2,\dots,u_n)\gamma u-\mathcal{M}(u,u_2,\dots,u_n)u\gamma u$=0 . … (7)
Let $\mathcal{A}(u)=\mathcal{M}(u^2,u_2,\dots,u_n)-\mathcal{M}(u,u_2,\dots,u_n)u$, then $\mathcal{A}(u)\gamma u$=0. … (8)
Replacing γ by $uz\mathcal{A}(u)$ in Equation (8) will get
$\mathcal{A}(u)uz\mathcal{A}(u)u$=0, hence $\mathcal{A}(u)u\mathcal{R}\mathcal{A}(u)u$=0. … (9)
Since $\mathcal{R}$ is a semiprime then $\mathcal{A}(u)u$=0. … (10)
Now, let $u=u+\gamma$ in Equation (10) we have $\mathcal{A}(u)u$+ $\mathcal{A}(\gamma)u$+ $\mathcal{A}(u)\gamma$+ $\mathcal{A}(\gamma)\gamma$=0. … (11)
Now, $\mathcal{A}(u+\gamma)$= $\mathcal{M}(u^2,u_2,\dots,u_n)-\mathcal{M}(u,u_2,\dots,u_n)u+ \mathcal{M}(y^2,u_2,\dots,u_n)-\mathcal{M}(y,u_2,\dots,u_n)\gamma$+ $\mathcal{M}(u\gamma+\gamma u,u_2,\dots,u_n)-\mathcal{M}(u,u_2,\dots,u_n)\gamma$-$\mathcal{M}(y,u_2,\dots,u_n)u$. … (12)
Let $\mathcal{B}(u,\gamma)$= $\mathcal{M}(u\gamma+\gamma u,u_2,\dots,u_n)-\mathcal{M}(u,u_2,\dots,u_n)\gamma$- $\mathcal{M}(y,u_2,\dots,u_n)u$. Then, we have from Equation (12)

$\mathcal{B}$(u,γ)+ $\mathcal{A}$(u)+ $\mathcal{A}$(γ), for all u, γ∈ $\mathcal{R}$.From Equation (11) implies that$\mathcal{B}$(u,γ)u +$\mathcal{A}$(u)u+ $\mathcal{A}$(γ)u+$\mathcal{B}$(u,γ)γ + $\mathcal{A}$(u)γ+ $\mathcal{A}$(γ)γ=0.

By Equation (10),$\mathcal{A}$(u)γ+ $\mathcal{B}$(u,γ)u +$\mathcal{A}$(γ)u+$\mathcal{B}$(u,γ)γ=0. … (13)

Now, let $u = -u$ in Equation (13) we get$\mathcal{A}$(u)γ+ $\mathcal{B}$(u,γ)u − $\mathcal{A}$(γ)u − $\mathcal{B}$(u,γ)γ=0. … (14)

Adding Equations (13) with (14) and using the fact that $\mathcal{R}$ is a 2-torsion free semiprime ring we find that$\mathcal{A}$(u)γ+ $\mathcal{B}$(u,γ)u=0. … (15)

Right multiplication of Equation (15) by $\mathcal{A}$(u)to get $\mathcal{A}$(u)γ$\mathcal{A}$(u)+ $\mathcal{B}$(u,γ)u$\mathcal{A}$(u)=0. … (16)

From Equation (10) we have,u$\mathcal{A}$(u)γu$\mathcal{A}$(u)=0. Then u$\mathcal{A}$(u) $\mathcal{R}$u$\mathcal{A}$(u)=0. … (17)

Also,u$\mathcal{A}$(u)=0. …(18)

From Equation (16) and by using Equation (18) will get$\mathcal{A}$(u)γ$\mathcal{A}$(u)=0 that $\mathcal{A}$(u) $\mathcal{R}\mathcal{A}$(u)=0, then $\mathcal{A}$(u)=0;
i.e. $\mathcal{M}(u^2, u_2, \dots, u_n) - \mathcal{M}(u, u_2, \dots, u_n)$u=0$\mathcal{M}(u^2, u_2, \dots, u_n)$=$\mathcal{M}(u, u_2, \dots, u_n)$u. Therefore, $\mathcal{M}$ is a Jordan left n- multiplier and $\mathcal{M}$ is a left n-multiplier on$\mathcal{R}$.

In Theorem 3.3, Substituting $y = u$, then we obtains the following:

Corollary 3.4: Let $\mathcal{R}$ be a 2-torsion free semiprime ring If$\mathcal{M}$:$\mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R} \rightarrow \mathcal{R}$ is a n-additive mapping such that $\mathcal{M}(u^3, u_2, \dots, u_n)$= $\mathcal{M}(u, u_2, \dots, u_n)u^2$ for all $u, u_2, \dots, u_n \in \mathcal{R}$. Then $\mathcal{M}$ is a symmetric left n-multiplier on$\mathcal{R}$.

Theorem 3.5:
Assume that $\mathcal{R}$ be a 2-torsion free semiprime ring and let $\mathcal{M}: \mathcal{R} \times \mathcal{R} \times \dots \times \mathcal{R} \rightarrow \mathcal{R}$ be an n-additive mapping. Suppose2$\mathcal{M}(uyu, u_2, \dots, u_n)$= $\mathcal{M}(u, u_2, \dots, u_n)yu + uy\mathcal{M}(u, u_2, \dots, u_n)$ holds for all$y, u, u_2, \dots, u_n \in \mathcal{R}$. In this case $\mathcal{M}$ is a n-multiplier."

Proof:

$2\mathcal{M}(uyu, u_2, \dots, u_n) = \mathcal{M}(u, u_2, \dots, u_n)yu + uy\mathcal{M}(u, u_2, \dots, u_n)$. … (1)

Substituting $u = u + z$ in Equation (1) to get
$2\mathcal{M}(uyz + zyu, u_2, \dots, u_n) =$
$\mathcal{M}(u, u_2, \dots, u_n)yz\mathcal{M}(z, u_2, \dots, u_n)yu$ +uy$\mathcal{M}(z, u_2, \dots, u_n) + zy\mathcal{M}(u, u_2, \dots, u_n)$. … (2)

Let $z = u^2$ in Equation (1) will get
$2\mathcal{M}(uyu^2 + u^2yu, u_2, \dots, u_n) = \mathcal{M}(u, u_2, \dots, u_n)yu^2 + \mathcal{M}(z, u_2, \dots, u_n)yu + uy\mathcal{M}(u^2, u_2, \dots, u_n) + u^2y\mathcal{M}(u, u_2, \dots, u_n)$. … (3)

Replacing $y = uy + yu$ in Equation (1) will get
$2\mathcal{M}(u^2yu + uyu^2, u_2, \dots, u_n) =$
$\mathcal{M}(u, u_2, \dots, u_n)uyu + \mathcal{M}(u, u_2, \dots, u_n)\, uyu^2 + u^2y\mathcal{M}(u, u_2, \dots, u_n)$ +uyu $\mathcal{M}(u, u_2, \dots, u_n)$. ...(4)

Subtracting Equations (4) from (3), we obtain($\mathcal{M}(u^2, u_2, \dots, u_n) - \mathcal{M}(u, u_2, \dots, u_n)$u)$yu$+($\mathcal{M}(u^2, u_2, \dots, u_n)$-u $\mathcal{M}(u, u_2, \dots, u_n) = 0$ for all $y, u, u_2, \dots, u_n \in \mathcal{R}$.

From the above relation and the Lemma2.1 we deduce
Ą$(u)yu = 0$, for all , $y \in \mathcal{R}$, where Ą(u) stands for $2\mathcal{M}(u^2, u_2, \dots, u_n) - \mathcal{M}(u, u_2, \dots, u_n)$u-u $\mathcal{M}(u, u_2, \dots, u_n)$. … (5)

Let uyĄ$(u) = y$ in Equation (5), then Ą$(u)\, uy$Ą$(u)u = 0$, $u, y \in \mathcal{R}$.Which implies thatĄ$(u)u = 0$, for all $\in \mathcal{R}$. … (6)

Multiply the left hand side of Equation (5) by u and from the right side by Ą(u) to get uĄ$(u)\, yu$Ą$(u)u = 0$ implies that uĄ$(u) = 0$. … (7)

Replacing $u = u + y$ in Equation (6) gives
Ą$(u)y$ + Ą$(y)u$ +$\mathcal{B}(u, y)u$ +$\mathcal{B}(u, y)y$ =0. Where $\mathcal{B}(u, y)$ stands for
$2\mathcal{M}(uy + yu, u_2, \dots, u_n) - \mathcal{M}(u, u_2, \dots, u_n)$y-$\mathcal{M}(y, u_2, \dots, u_n)$-$\boldsymbol{x}\mathcal{M}(y, u_2, \dots, u_n) - y\mathcal{M}(u, u_2, \dots, u_n)$.

Replacing -$u = u$ in the last relation and combining the new relation with the old one, will get Ą$(u)y + \mathcal{B}(u, y)u = 0$. … (8)

Multiply the right hand side of Equation (8) by Ą(u) and using Equation (7) we obtainĄ$(u)y$ Ą$(u) = 0$, which implies that Ą$(u) = 0$,follows. Therefore
$2\mathcal{M}(u^2, u_2, \dots, u_n) = \mathcal{M}(u, u_2, \dots, u_n)$u + u $\mathcal{M}(u, u_2, \dots, u_n)$. … (9)

The result $\mathcal{M}(u^2, u_2, \dots, u_n) = \mathcal{M}(u, u_2, \dots, u_n)$u and $\mathcal{M}(u^2, u_2, \dots, u_n) =$ u $\mathcal{M}(u, u_2, \dots, u_n)$ for all $y, u, u_2, \dots, u_n \in \mathcal{R}$ by Proposition 3.2. However, we shall give an immediate proof as follows. We intend to prove that $[\mathcal{M}(u, u_2, \dots, u_n), u] = 0$. … (10)

The linearization of Equation (9) gives
$2\mathcal{M}(uy + yu, u_2, \dots, u_n) = \mathcal{M}(u, u_2, \dots, u_n)y + \mathcal{M}(y, u_2, \dots, u_n)u + u\,\mathcal{M}(y, u_2, \dots, u_n) + y\,\mathcal{M}(u, u_2, \dots, u_n)$. …(11)

Replacing $y = 2uyu$ in Equation (11), will get, according to the hypotheses $4\,\mathcal{M}(u^2yu + uyu^2, u_2, \dots, u_n)$ =2 $\mathcal{M}$ (u, $u_2, \dots, u_n$) uyu + $\mathcal{M}$ (u,$u_2, \dots, u_n$,)y u^2 + $uy\mathcal{M}$ (u, $u_2, \dots, u_n$) u + $u\mathcal{M}$(u,$u_2, \dots, u_n$)yu+ u^2 y $\mathcal{M}$(u,$u_2, \dots, u_n$)+$2uyu\mathcal{M}(u, u_2, \dots, u_n)$. … (12)

By comparing Equations (4) with (12), will get $\mathcal{M}$ (u,$u_2, \dots, u_n$,)y u^2 + u^2 y $\mathcal{M}$ (u, $u_2, \dots, u_n$)-uy$\mathcal{M}$(u,$u_2, \dots, u_n$)u - u$\mathcal{M}$(u,$u_2, \dots, u_n$)yu=0, for all , $y, u, u_2, \dots, u_n \in \mathcal{R}$. …(13)

Let $u = yu$ in the Equation (13), will get

$\mathcal{M}$(u,$u_2, \dots, u_n$)yu^3+ u^2 y $u\mathcal{M}$(u,$u_2, \dots, u_n$)- uyu$\mathcal{M}$(u,$u_2, \dots, u_n$)u - u$\mathcal{M}$(u,$u_2, \dots, u_n$)yu^2=0. …(14)

Multiply the right hand side of Equation (13) by u to get $\mathcal{M}$(u,$u_2, \dots, u_n$)yu^3+u^2 y $\mathcal{M}$(u,$u_2, \dots, u_n$)u-uy $\mathcal{M}$(u,$u_2, \dots, u_n$)u^2- u$\mathcal{M}$(u,$u_2, \dots, u_n$)yu^2=0. …(15)

Subtracting Equations (14) from (15), will arrive atu^2 y [$\mathcal{M}$(u,$u_2, \dots, u_n$),u]- uy [$\mathcal{M}$(u,$u_2, \dots, u_n$),u]u=0. …(16)

Replacing $y = \mathcal{M}$(u,$u_2, \dots, u_n$)y in Equation (16) will get

$u^2\mathcal{M}$(u,$u_2, \dots, u_n$)$y[\mathcal{M}(u, u_2, \dots, u_n), u] - u\mathcal{M}$(u,$u_2, \dots, u_n$)y $[\mathcal{M}(u, u_2, \dots, u_n), u]u = 0$for all $y, u, u_2,$ $\dots, u_n \in \mathcal{R}$.…(17)

Multiply the right hand side of Equation (16) by $\mathcal{M}$ (u,$u_2, \dots, u_n$,to get $\mathcal{M}$(u,$u_2, \dots, u_n$)u^2 y [$\mathcal{M}$(u,$u_2, \dots, u_n$),u]-$\mathcal{M}$(u,$u_2, \dots, u_n$)uy [$\mathcal{M}$(u,$u_2, \dots, u_n$)$, u]u$=0. … (18)

When subtracting Equations (17) from (18) we get [$\mathcal{M}$ (u,$u_2, \dots, u_n$), u^2]y[$\mathcal{M}$ (u,$u_2, \dots, u_n$),u]-[$\mathcal{M}$(u,$u_2, \dots, u_n$),$u]y[\mathcal{M}$(u,$u_2, \dots, u_n$),$u]u = 0$ for all , $y, u, u_2, \dots, u_n \in \mathcal{R}$.

According to the Lemma (2.1)can replace the above relation by

([$\mathcal{M}$ (u,$u_2, \dots, u_n$), u^2]-[$\mathcal{M}$ (u,$u_2, \dots, u_n$)$, u]u)[\mathcal{M}$ (u,$u_2, \dots, u_n$)$, u] = 0$ which is equivalent to $u[\mathcal{M}$(u,$u_2, \dots, u_n$),$u]$ $y[\mathcal{M}$(u,$u_2, \dots, u_n$),u]=0.

Let $y = yu$ in the last relation, to get $[\mathcal{M}$ (u,$u_2, \dots, u_n$)$, u]$ $[\mathcal{M}$ (u,$u_2, \dots, u_n$)$, u] = 0$. Then, we have [$\mathcal{M}$(u,$u_2, \dots, u_n$),u]=0. …(19)

Of course we have also [$\mathcal{M}$(u,$u_2, \dots, u_n$)$, u] = 0$. … (20)

From Equation (19) one obtains (see how (8) has been obtained from (6)) u [$\mathcal{M}$ (u,$u_2, \dots, u_n$)$, y]$ + $u[\mathcal{M}$(u,$u_2, \dots, u_n$),u]+ y[$\mathcal{M}$(u,$u_2, \dots, u_n$)$, u] = 0$.

Multiply the left hand side of the last Equation by [$\mathcal{M}$(u,$u_2, \dots, u_n$)$, u$] and using Equation (20), to get

[$\mathcal{M}$(u,$u_2, \dots, u_n$),$u]y$ [$\mathcal{M}$(u,$u_2, \dots, u_n$),u] =. Since $\mathcal{R}$ is a simeprime ring, the result is [$\mathcal{M}$(u,$u_2, \dots, u_n$)$, u] =$ 0.

Combining Equations (9) with (10) we get

$\mathcal{M}(u^2, u_2, \dots, u_n) = \mathcal{M}$(u,$u_2, \dots, u_n$)u for all $u, u_2, \dots, u_n \in \mathcal{R}$ and $\mathcal{M}(u^2, u_2, \dots, u_n)$= $u\mathcal{M}$(u,$u_2, \dots, u_n$) which means that $\mathcal{M}$ is a left and also a right Jordan n-multiplier. By Proposition 3.2 $\mathcal{M}$ is n-multiplier.

Theorem 3.6: Let $\mathcal{R}$ be a prime ring with char ($\mathcal{R}$) ≠ 2 and U a nonzero Lie ideal of$\mathcal{R}$. If $\mathcal{R}$ admits a nonzero left n-multiplier$\mathcal{M}$:$\mathcal{R} \times \mathcal{R} \times \dots \times \mathcal{R} \to \mathcal{R}$ such that $\mathcal{M}(u, u_2, \dots, u_n)$ $\mathcal{M}(y, u_2, \dots, u_n)$-$\mathcal{M}(uy, u_2, \dots, u_n)$ ∈Z($\mathcal{R}$) for all y,$u, u_2, \dots, u_n$∈U, then either U⊆$Z(\mathcal{R})$ or $\mathcal{M}$(r,$u_2, \dots, u_n$) = r for all $r, y, u, u_2, \dots, u_n$∈$\mathcal{R}$.

Proof:

Assume that $U \not\subset Z(\mathcal{R})$, we have by hypothesis$\mathcal{M}(u, u_2, \dots, u_n)\mathcal{M}(y, u_2, \dots, u_n) - \mathcal{M}(uy, u_2, \dots, u_n) \in Z(\mathcal{R})$. …(1)

Let $y = [y, z]$ in Equation (1), this leads to $Z(R) \ni \mathcal{M}(u, u_2, \dots, u_n)\mathcal{M}([y, z], u_2, \dots, u_n)$- $\mathcal{M}(u[y, z], u_2, \dots, u_n)$ = $\mathcal{M}$ ($u, u_2, \dots, u_n$) $\mathcal{M}$ ($yz, u_2, \dots, u_n$)- $\mathcal{M}$ ($u, u_2, \dots, u_n$) $\mathcal{M}$ ($zy, u_2, \dots, u_n$) - $\mathcal{M}$ ($uyz, u_2, \dots, u_n$) + $\mathcal{M}(uzy, u_2, \dots, u_n)$ ∈$Z(\mathcal{R})$.

For all $y, z, u, u_2, u_3, \dots, u_n$∈$U$ and $\mathcal{M}$ is a left n- multiplier, then will get

= ($\mathcal{M}$ ($u, u_2, \dots, u_n$) $\mathcal{M}$ ($y, u_2, \dots, u_n$)- $\mathcal{M}$ ($uy, u_2, \dots, u_n$)) z _ ($\mathcal{M}$ ($u, u_2, \dots, u_n$) $\mathcal{M}$ ($z, u_2, \dots, u_n$) _ $\mathcal{M}(uz, u_2, \dots, u_n)$)$y$∈$Z(\mathcal{R})$. …(2)

Commuting Equation (2) with z to get

($\mathcal{M}$ ($u, u_2, \dots, u_n$)$\mathcal{M}$ ($y, u_2, \dots, u_n) - \mathcal{M}(uy, u_2, \dots, u_n)[z, z]$ -($\mathcal{M}$ ($u, u_2, \dots, u_n$) $\mathcal{M}$ ($z, u_2, \dots, u_n$) - $\mathcal{M}(uz, u_2, \dots, u_n)[y, z]$=0.

Now, Let $y = wy$ in above Equation

($\mathcal{M}$ ($u, u_2, \dots, u_n$) $\mathcal{M}$ ($z, u_2, \dots, u_n$) _ $\mathcal{M}$ ($uz, u_2, \dots, u_n$)) $w[y, z$]=0. Since $\mathcal{R}$ is a prime ring, then either($\mathcal{M}(u, u_2, \dots, u_n)\mathcal{M}(z, u_2, \dots, u_n)$ - $\mathcal{M}(uz, u_2, \dots, u_n)$) = 0 or $[y, z]$ =0.

Suppose thatA_1,A_2 both are n-additive subgroups of U.

A_1= {z∈U| $\mathcal{M}(u, u_2, \dots, u_n)\mathcal{M}(z, u_2, \dots, u_n)$ - $\mathcal{M}(uz, u_2, \dots, u_n)$= 0 for all $u, u_2, u_3, \dots, u_n$∈U}

So A_2={z∈U|$[y, z]$=0 for all $y \in U$} ,then $A_1 \cup A_2 = U$, Since a group cannot be the union of its two proper subgroups follows that A_2=U or A_2=U. Then $\mathcal{M}(u, u_2, \dots, u_n)\mathcal{M}(z, u_2, \dots, u_n)$ - $\mathcal{M}(uz, u_2, \dots, u_n)$= 0 or [U, U]=0, by Lemma 2.5we have $U \subseteq Z$ ($\mathcal{R}$), But this a contraction ,thus we can assume that, $\mathcal{M}(u, u_2, \dots, u_n)\mathcal{M}(z, u_2, \dots, u_n) - \mathcal{M}(uz, u_2, \dots, u_n) = 0$. Since $\mathcal{M}$ is a left n -multiplier of $\mathcal{R}$, then $\mathcal{M}(u, u_2, \dots, u_n)(\mathcal{M}(z, u_2, \dots, u_n) - z) = 0$. ....(3)

Let $z = [z, r]$ in Equation (3) to get
$\mathcal{M}(u, u_2, \dots, u_n)$ ($\mathcal{M}$(z, $u_2, \dots, u_n$)-z)r - $\mathcal{M}(u, u_2, \dots, u_n)$($\mathcal{M}(r, u_2, \dots, u_n)$-$r$)$z$ =0. By Equation (3) we get$\mathcal{M}(u, u_2, \dots, u_n)$($\mathcal{M}$(r, $u_2, \dots, u_n$)-r) z=0
By Lemma 2.4, $\mathcal{M}(u, u_2, \dots, u_n)(\mathcal{M}(\mathrm{r}, u_2, \dots, u_n)\text{-}r) = 0.$ … (4)
Left multiplying in Equation (4) by $\mathcal{M}(y, u_2, \dots, u_n)$ to get,
$\mathcal{M}(y, u_2, \dots, u_n)u(\mathcal{M}$ (r, $u_2, \dots, u_n$)-r)=0.
By Lemma 2.4 and for all $r \in \mathcal{R}$, $U \in \mathcal{R}$, $r, u, u_2, u_3, \dots, u_n \in U$,we have $\mathcal{M}(y, u_2, u_3, \dots, u_n)\mathcal{R}(\mathcal{M}$(r, $u_2, u_3, \dots, u_n$)-r)=0. … (5)
Putting $y = [u, r]$ in Equation (5) to get the following $\mathcal{M}$([u, r], $u_2, \dots, u_n$)=$\mathcal{M}(u, u_2, \dots, u_n)r - \mathcal{M}$(r, $u_2, \dots, u_n$)u=0.
If $\mathcal{M}(u, u_2, \dots, u_n) = 0$, then $\mathcal{M}$(r, $u_2, \dots, u_n$) u=0 for all $u, u_2, \dots, u_n \in U$, that is $\mathcal{M}(\mathcal{R}, u_2, \dots, u_n)U$=0 and by Lemma2.4,$\mathcal{M}(\mathcal{R}, u_2, \dots, u_n) = 0$, which is a contradiction.

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