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A Case Study on 3dimensional Shapes of Curves and Its Application

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Abstract: In this paper, we are going to study about the 3-dimensional curves and its application. The curves come under the topic Analytical Geometry. The purpose of this paper is to show how the 3-dimensional curves are applied in real life and also what are the benefits of using these types of curves. The curves discussed are helix, asteroid, cycloid and solenoid. The curves are investigated using its parametric equations. Some applications of these curves are DNA, Roller coaster, pendulum, electro-magnetic wires, magnetic field, etc. are illustrated with example problems. A detailed mathematical analysis of 3-d curves is discussed below.
Keywords: Asymptote, Axis, Cartesian co-ordinates, Curvature, Vertex

I. INTRODUCTION

In Analytical Geometry, a curve is generally a line, in which its curvature has not been zero. A Curve is also called as a curved line, it is similar to a line, but it need not be a straight line. Some examples of these curves are parabola, ellipse, hyperbola, etc. The curves are classified into two types. They are (i) Open curve- Open curve is defined as a curve whose end does not meet. For example, parabola, hyperbola. (ii) Closed curve- Closed curve is defined as the curves whose ends meet. The closed curves do not have end points. For example, ellipse, sphere. In these two types of curves we have many branches of curves. The curves are divided based on their shapes.

In 3D curves, the important technical terms used are locus, focus, vertex, latus rectum, asymptote, oblique asymptote. Locus -A locus is a set of all points (commonly, a line, a line segment, a curve or a surface), whose location satisfies or is determined by one or more specified conditions. Focus- A focus, are special points with reference to which any of a variety of curves is constructed. For example, one or two foci can be used in defining conic sections, the four types of which are the circle, ellipse, parabola, and hyperbola. In addition, two foci are used to define the Cassini oval and the Cartesian oval, and more than two foci are used in defining an n-ellipse. Vertex- A vertex is a point where two or more curves, lines, or edges meet. As a consequence of this definition, the point where two lines meets to form an angle and the corners of polygons and polyhedral are vertices. Latus rectum-The latus rectum of a conic section is the chord (line segment) that passes through the focus is perpendicular to the major axis and has both endpoints on the curve. The latus rectum of a conic section is the chord through a focus parallel to the conic section directrix. "Latus rectum" is a compound of the Latin word latus, meaning "side," and rectum, meaning “straight”. Half of the latus rectum is called the semi-latus rectum. Asymptote -An asymptote of a curve is a line such that the distance between the curve and the line approaches zero as one or both of the x or y coordinates tends to infinity. Some sources include the requirement that the curve may not cross the line infinitely often, but this is unusual for modern authors. In some contexts, such as algebraic geometry, an asymptote is defined as a line which is tangent to a curve at infinity. More generally, one curve is a curvilinear asymptote of another (as opposed to a linear asymptote) if the distance between the two curves tends to zero as they tend to infinity, although the term asymptote by itself is usually reserved for linear asymptotes. The area of analytical geometry is a very broad field of study. It is used to model the exact phenomena that are used in real life.

The basic concepts of 3D curves , its representations are described in[2] by E. Bribiesca. A. Jonas and N. Kiryati [4] discussed the digital representation schemes for 3D curves. The representation of 3D segments and its constructions are discussed in [5] and [6]. I. Arjuna [1] explained the parametric equation and the Cartesian co-ordinates of the cycloid curve. In [3], A J.Emmerson describes about how the pendulum works by using cycloid curves.

This paper is outlined as follows: In section 2, we have given some basic definitions of 3D curves and some mathematical description. Section 3 interprets the application problems of 3D curves.

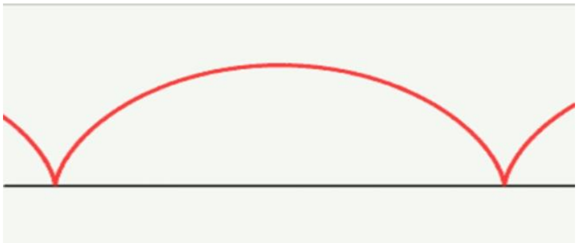
II. PRELIMINARIES AND EQUATION

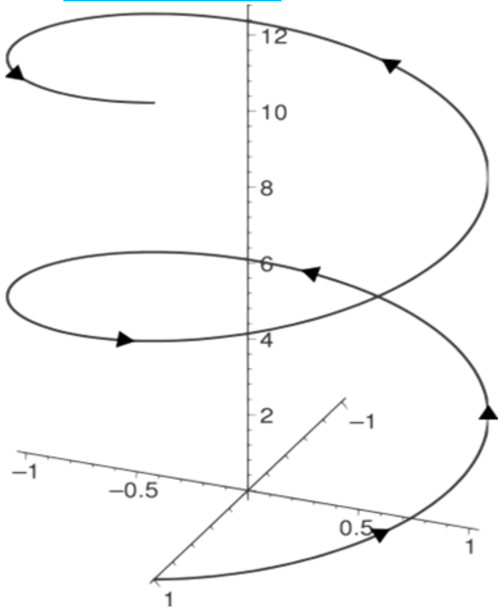
Basic definition regarding the 3D shapes of curves, steady states and the mathematical equation is presented in this section.

Definition: 1 Helix curve- A helix is a type of smooth space curve, i.e. a curve in three dimensional spaces. It has the property that the tangent line at any point makes a constant angle with a fixed line called the axis. Example: Coil springs, spiral staircases

Helix Curve

Cycloid curve





Mathematical Description: The helix is a space curve with parametric equations

$x = r\cos t$------------------------ (1.1)

$y = r\sin t$------------------------ (1.2)

$z = c\, t$ ------------------------- (1.3)

For $t \varepsilon\ [0,2\pi]$, where r is the radius of the helix and $2\pi c$ is a constant giving the vertical separation of the helix's loops.

The curvature of the helix is given by

$k = \frac{r}{r^2+c^2}$ ------------------------- (1.4)

and the locus of the centers of curvature of a helix is another helix. The arc length is given by

$r = \sqrt{r^2+c^2}\ t$--------------------- (1.5)

The torsion of a helix is given by

$\tau = \frac{c}{r^2+c^2}$ --------------------------- (1.6)

Equations (4/5), we get

$\frac{k}{\tau} = \frac{r}{c}$ ------------------------------ (1.7)

which is constant. In fact, Lancret's theorem states that a necessary and sufficient condition for a curve to be a helix is that the ratio of curvature to torsion be constant.

Definition: 2 Cycloid curves- A cycloid is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slipping. A cycloid is a specific form of "trochoid" and is an example of roulette, a curve generated by a curve rolling on another curve. The cycloid, with the cusps pointing upward, is the curve of fastest descent under constant gravity, and is also the form of a curve for which the period of an object in descent on the curve does not depend on the object's starting position.

Mathematical Description: By using Huygen's Law,
The parametric equations of the cycloid are

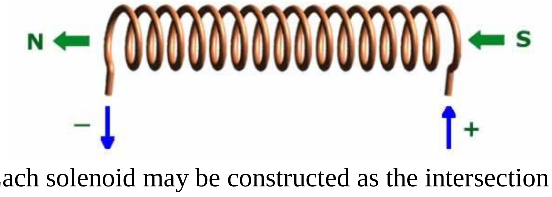
$x = a(\theta - \sin\theta)$ --------------------------- (2.1)

$y = a(1 - \cos\theta)$ --------------------------- (2.2)

To see that the cycloid satisfies the tautochrone property, consider the derivatives

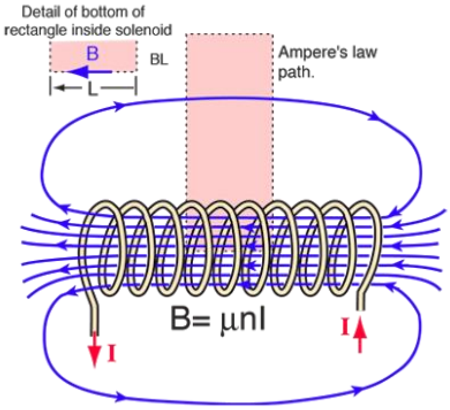
Differentiate the equation (1), (2) with respect to θ,
$x' = a(1 - \cos\theta)$ --------------------------- (2.3)
$y' = a\sin\theta$ ------------------------------------ (2.4)
and
$x'^2 + y'^2 = a^2[(1 - 2\cos\theta + \cos^2\theta) + \sin^2\theta]$
$= 2a^2(1 - \cos\theta)$----------------- (2.5)
Now
$\frac{1}{2}mv^2 = mgy$ --------------------------------- (2.6)
$v^2 = \frac{2mgy}{m}$
$V = \frac{ds}{dt} = \sqrt{2gy}$ ----------------------- ----- (2.7)
$dt = \frac{ds}{\sqrt{2gy}}$ --------------------------------(2.8)
$= \frac{\sqrt{x' + y'}}{\sqrt{2gy}}$
$= \frac{a\sqrt{2(1-\cos\theta}}{\sqrt{2ga(1-\cos\theta}} d\theta$
$= \sqrt{\frac{a}{g}} d\theta$ ------------------------------ (2.9)
So that time required to travel from the top of the cycloid to the bottom is
$T = \int_0^\pi dt = \sqrt{\frac{a}{g}}\pi$ -------------------- (2.10)
However, from an intermediate point θ_0,
$v = \frac{ds}{dt} = \sqrt{2g(y - y_{0)}}$ ---------------- (2.11)
so,
$T = \int_{\theta_0}^{\pi} \sqrt{\frac{2a^{2(1-\cos\theta)}}{2ag(\cos\theta_0 - \cos\theta}} d\theta$
$= \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \sqrt{\frac{1-\cos\theta}{\cos\theta_{0-\cos\theta}}} d\theta$ ----------------- (2.12)
Integrate this equation using the half-angle formulas, we obtain
$T = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \frac{\sin\left(\frac{1}{2}\theta\right)d\theta}{\cos^2\left(\frac{1}{2}\theta_0\right) - \cos^2\left(\frac{1}{2}\theta\right)}$ ------------ (2.13)
Now transform variables to
$u = \frac{\cos\left(\frac{1}{2}\theta\right)}{\cos\left(\frac{1}{2}\theta_0\right)}$,
$du = -\frac{\sin\left(\frac{1}{2}\theta\right)d\theta}{2\cos\left(\frac{1}{2}\theta_0\right)}$ ------------------------- (2.14)
so we obtain
$T = -2\sqrt{\frac{a}{g}} \int_1^0 \frac{du}{\sqrt{1-u^2}}$-
$= 2\sqrt{\frac{a}{g}}[\sin^{-1} 0 - \sin^{-1} 1] = \pi\sqrt{\frac{a}{g}}$---------------------------------- (2.15)
and the amount of time is the same from any point.

Definition: 3- Solenoid Curve- A solenoid is a compact connected topological space that may be obtained as a inverse limit of an inverse system of topological groups and continuous homomorphism.



Mathematical description: Each solenoid may be constructed as the intersection of a nested system of embedded solid tori inR^3.
Fix a sequence of natural numbers$\{n_i\}$, $n_i \geq 2$.

Le $T_o = S^1 \times D$ is a solid torus. For each $i \geq 0$, choose a solid torus T_{i+1} that is wrapped longitudinally n_i times inside the solid torus T_i.
Then their intersection
$\wedge = \bigcap_{i\geq 0} T_i$ ------------------ (3.1)
is homeomorphism to the solenoid constructed as the inverse limit of the system of circles with the maps determined by the sequence $\{n_i\}$.
For the magnetic field in solenoid curve,



n = N/L, ----------------------- (3.2)
is the number of turns per unit length.
The magnetic field is directly proportional to the current I in the solenoid coil.
By using Ampere's Law,
$BL = \mu NI$
$B = \mu \frac{N}{L} I$
$B = \mu nI$ ------------------------ (3.3)
By using this equation, we get the number of turns and also a magnetic field in solenoid curve.

Definition 4: Hooke's law: It is a principle of physics that states that the force (F) needed to extend or compress a spring by some distance X scales linearly with respect to that distance. That is: $F = kX$, where k is a constant factor characteristic of the spring: its stiffness, and X is small compared to the total possible deformation of the spring

Definition 5: Huygens's Principle: Every point on a propagating wavefront serves as the source of spherical secondary wavelets, such that the wavefront at some later time is the envelope of these wavelets. If the propagating wave has a frequency, f, and is transmitted through the medium at a speed, v, then the secondary wavelets will have the same frequency and speed.
This principle is quite useful, for from it can be derived the laws of reflection and refraction

III. APPLICATION PROBLEMS

1. Find the work required to compress a spring from its natural length of 1 foot to a length of 0.50 foot if the force constant is k=16 kg/foot.

Solution:

"Hooke's law" ways that the force it takes to stretch or compress a spring x length units from its natural length is proportional to x.

The formula of Hooke's law is represented as F=kx, where F is the force, k is the spring constant, and x is the extension of the material

Suppose the given spring is placed on the x-axis. It is fixed at x=1 and movable and at the origin. From the above formula, the force is required to compress the spring from 0 to x with the formula, F=16x.

To compress the spring from 0 to 0.25ft, the force must increase from F (0) = 0 to
F (0.50) = 16*0.50 = 8 foot.kg

Therefore the work done by F over the interval is,
$W = \int_0^{0.50} 16x dx = 2.0 \text{ ft. kg.}$

2. A roadway goes from tangent alignment to a 250 m circular curve by means of a 80m long spiral transition curve. The angle of deflection between the tangents is 45 degree. Using the formulas compute X_S, Y_S, P and

K. Assuming that the station of the P.L measured along the back tangent, is 250+00, and compute the stations of the TS, SC, CS and ST.

Solution:

Determine spiral angle and co-ordinates of SC points

R_C=250m L_S=80m $Q_S = {R_C}/{L_S} = \frac{80}{2(250)} = 0.160$ rad

A=$\sqrt{(R_C L_S)}$=$\sqrt{(80 \times 250)}$=141.42

$$X_S = L_S - \frac{L_S^5}{40A^4} + \frac{L_S^9}{3456A^8}$$

$$= 80 - \frac{80^5}{40(141.42^4)} + \frac{80^9}{3456(141.42^8)} = 79.795\text{m}$$

$$Y_S = \frac{L_S}{6A^2} - \frac{L_S^7}{336A^6} + \frac{L_S^{11}}{42240A^{10}}$$

$$= \frac{80^3}{6(141.42^2)} - \frac{80^7}{336(141.42^6)} - \frac{80^{11}}{42240(141.42^{10})} = 4.259\text{m}$$

Determine P,K, T' andL_C

$P = Y_S - R(1 - \cos\theta_S) = 4.259 - 250[1 - \cos(0.160\text{rad})] = 1.066\text{m}$

$K = X_S - R_C \sin\theta_s = 79.795 - 250\sin(0.160\text{rad}) = 39.965\text{m}$

$T' = (R_C + p)\tan(\frac{\Delta}{2}) = (250 + 1.066)\tan(\frac{45^\circ}{2}) = 103.995\text{m}$

$L_C = R_C\Delta_{rad} - L_S = 250(0.785) - 80 = 116.250\text{m}$

TS station $= \text{P.I.station} - (T' + k) = (250 + 00) - [(1 + 04) + (0 + 40)] = 248 + 56$

SC Station $= \text{TSstation} + L_S = (248 + 56) + (0 + 80) = 249 + 36$

CS station $= \text{SCstation} + L_C = (249 + 56) + (1 + 16) = 250 + 52$

ST station $= \text{CSstation} + L_S = (250 + 52) + (0 + 80) = 251 + 32$

3. What is the frequency of a cycloidal pendulum with length of 28"?

Solution:

Given: L = 28", f =?

First, convert the length into meters

$$L = 28'' = 71.12\text{cm} = 0.7112\text{m}$$

Next, plug the length and acceleration due to gravity into Huygen's Law.

$$T = 2\pi\sqrt{\frac{L}{g}}$$

We know the gravity on surface on earth =9.8m/s

$$T = 2\pi\sqrt{\frac{(0.7112)}{(9.80)}}$$

$= 2\pi(0.2694) = 1.69$ s

Now that we have the period T, we take the reciprocal to find the frequency.

$f = \frac{1}{T} = \frac{1}{1.69s} = 0.5917$HZ

4. What is the length of a cycloid pendulum with a period of 20 seconds?

Solution:

Given:T = 20.0s

Unknown: L =?

First, we take Huygen's Law: $T = 2\pi\sqrt{\frac{L}{g}}$

And we algebraically isolate the variable L. First, we square both sides of the equation and cross-multiply, to get

$T^2 = 4\pi^2\left(\frac{L}{g}\right) \Rightarrow \frac{T^2 g}{4\pi^2} = L$

Then we plug in our values for T and g, to get $\frac{20^2 g}{4\pi^2} = L$

$$\frac{400 * 9.8}{4 * 3.14 * 3.14} = L \Rightarrow 99.4 = L$$

∴ The length of the cycloid pendulum is 99.4

5) Find the magnetic field produced by the solenoid if the number of loop is 500 and current passing through it is 5A. (Length of the solenoid is 40cm and $k = 10^{-7} n/Amps^2$)

Solution:

N= 500, i=5A, I=40cm =0.4m, k= $10^{-7} n/Amps^2$, l = 40 cm =0.4m

$$B = 4\pi k \frac{i.N}{l} = 4 * 3.14 \frac{10^{-7} * 5 * 500}{0.4}$$

$$B = 7.85 * 10^{-3} \frac{N}{Amps.m}$$

The magnetic field produced by using the solenoid wire we get 7.85*$10^{-3} \frac{N}{Amps.m}$

6) A solenoid has a diameter of 80cm, the number of loops is 4 and magnetic inside it is 1, 2.$\frac{10^{-5}N}{Amp}$. m. Find the current passing through the each loop of the wire.

Solution:

Since the question asks current on each loop, we assume each loop as circle, thus we find the magnetic field;

$$i_{loop} = n.i$$

$$r = \frac{80}{2} = 40cm = 0.4m$$

$$B = 2.3.10^{-7} \frac{0.4.i}{0.4}$$

$$i = 2Amps$$

IV. CONCLUSION

As a conclusion, it can be summarized that these curves - helix, cycloid, solenoid are having applications in various fields of engineering. In future work it is proposed to study the helical antennas which are ideal for amateur satellite communications.

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