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The Non Central Graph of Finite Groups

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Abstract: Let G be a finite non-abelian group. The non-central graph denoted by $\Gamma^{Z(G)}$ is a graph whose set of vertices is the non-central elements of G. A pair of vertices v_1 and v_2 are adjacent in the non- central graph whenever v_1 and v_2 do not belong to the center of G. In this paper, the non-central graph is determined for some finite groups. Besides, some graph properties are found.
Keywords: Graph theory; groups; center groups.

1. Introduction

Throughout this paper, Γ denotes a simple undirected graph. In the following we state a brief history about graph theory followed by some fundamental concepts in graph theory.

The concept of graph theory was firstly introduced in 1736 by Leonard Euler who solved Konigsberg bridge problem by drawing a graph with vertices and edges [1]. After some years, the usefulness of graph theory has been proven to a large number of devise fields.

The followings are some basic concepts of graph theory that may use in this paper. These concepts can be found in one of the references ([1], [2]).

A graph Γ is a mathematical structure consisting of two sets vertices and edges which are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. The graph is called directed if its edges are identified with ordered pair of vertices. Otherwise, Γ is called in-directed. Two vertices are adjacent if they are linked by an edge. A complete graph is a graph where each ordered pair of distinct vertices are adjacent, denoted by K_n [1, 2].

The followings are some basic concepts related to graph properties that are needed in this article.

A non-empty set S of $V(\Gamma)$ is called an independent set of Γ if there is no adjacent between two elements of S in Γ. Meanwhile, the independent number is the number of vertices in the maximum independent set and it is denoted by $\alpha(\Gamma)$. However, the maximum number c for which Γ is c-vertex colorable is known as chromatic number, denoted by $\chi(\Gamma)$. The diameter is the maximum distance between any two vertices of Γ, denoted by $d(\Gamma)$. Furthermore, a clique is a complete subgraph in Γ, while the clique number is the size of the largest clique in Γ and is denoted by $\omega(\Gamma)$. The dominating set $X \subseteq V(\Gamma)$ is a set where for each v outside X, there exists $x \in X$ such that v is adjacent to x. The minimum size of X is called the dominating number and it is denoted by $\gamma(\Gamma)$ ([1], [2], [3]).

The following theorem illustrates the center of the dihedral groups.

Theorem 1.1 [4, 5, 6] Let G be a dihedral group, $G \cong \langle a, b : a^n = b^2 = e, ab = ba \rangle$. Then

$$Z(G) = \begin{cases} \left\{1, a^{\frac{n}{2}}\right\}, & \text{if } n \text{ is even,} \\ \{1\}, & \text{if } n \text{ is odd.} \end{cases}$$

2. Results and Discussion

In this section, a new graph namely the non-central graph is introduced. The graph is found for dihedral groups, symmetric groups and alternating groups. In addition, some graph properties are obtained.

Definition 2.1 Let G be a finite non abelian group and $\Gamma^{Z(G)}$ is the non-central graph. The set $V\left(\Gamma^{Z(G)}\right)$ is the set of all non- central elements of G in which two vertices v_1 and v_2 are adjacent if they do not belong to $Z(G)$.

In the following context, we introduce our main results.

Theorem 2.1 Let G be a finite abelian group. Then $\Gamma^{Z(G)}$ is null.
Proof: The proof is clear. ■

Theorem 2.2 Let G be a finite dihedral group D_{2n}, $G \cong \langle a,b : a^n = b^2 = e, ab = ba \rangle$. Then

$$\Gamma^{Z(G)} = \begin{cases} K_{(2n-2)}, & if\ n \text{ is even,} \\ K_{2n-1}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Based on Theorem 1.1, when n is even, then the non-central graph $\Gamma^{Z(G)}$ consists one complete component of K_{2n-2}. In the case that n is odd, $\Gamma^{Z(G)}$ consists of one complete component of K_{2n-1}. The proof then follows. ■

Based on Theorem 2.2, the following corollary can be concluded.

Corollary 2.1 If G be a finite dihedral group D_{2n}, $G \cong \langle a,b : a^n = b^2 = e, ab = ba \rangle$ and $\Gamma^{Z(G)} = K_{2n-2}$ when n is even and equals $K_{2n\text{-}1}$ when n is odd, then

$$\chi\left(\Gamma^{Z(G)}\right) = \omega\left(\Gamma^{Z(G)}\right) = \begin{cases} 2n-2, & \text{if } n \text{ is even,} \\ 2n-1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof: Based on Theorem 2.2. The proof is thus straightforward. ■

In the following, the non-central graph is computed for the symmetric groups and alternating groups.

Theorem 2.3 Let G be a finite symmetric groups, S_n. Then $\Gamma^{Z(G)} = K_{(n!-1)}$.

Proof: Since the only element in $Z(S_n)$ is the identity and based on Definition 2.1 thus $\Gamma^{Z(G)}$ consists of one complete graph of $K_{(n!-1)}$. The proof is thus completed. ■

Corollary 2.2 If G be a finite symmetric group and $\Gamma^{Z(G)} = K_{(n!-1)}$, then $\chi\left(\Gamma^{Z(G)}\right) = \omega\left(\Gamma^{Z(G)}\right) = 2^{n-1}$.

Proof: Based on Theorem 2.3, there is only one complete graph of $K_{(n!-1)}$, thus there is (n!-1) colored vertices and cliques numbers. The proof is thus straightforward. ■

Theorem 2.4 Let G be a finite alternating group, A_n. Then $\Gamma^{Z(G)} = K_{\left(\frac{n!}{2}-1\right)}$.

Proof: The proof is similar to Theorem 2.3. ■

Theorem 2.5 Let G be a finite quaternion group Q_n, $G \cong \langle a,b : a^{2^{(n-1)}} = 1, bab^{-1} = a^{-1}, b^2 = a^{2(n-2)} \rangle$, $n \geq 3$. Then $\Gamma^{Z(G)} = K_{\left(2^{n-1}-2\right)}$.

Proof: Since the elements $\left\{1, a^{2^{(n-1)}}\right\} \in Z\left(Q_n\right)$ and based on Definition 2.1, thus $\Gamma^{Z(G)}$ contains a complete graph of $K_{\left(2^{n-1}-2\right)}$. The proof then follows. ■

Conclusion

In this paper, a new graph was introduced, namely the non-central graph. The graph was found for some finite non abelian groups. Besides, some graph properties were obtained.

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