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On Abelian Multiset Group

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Abstract: This paper seeks to further the work of Tripathy et al ([16]). The work is rooted and build from their new definition of Multi-Group (Multiset Group). We began with the establishment of the synergy and comparison between the Nazmul et al's([6]) and Tripathy et al's([16]) definition of multi-group. Where we discover that every multigroup is a multiset group but the converse need not hold we also study the generalisation of the closure of intersection of two or more multiset groups under multiset operation in which it is also a multiset group while that of union need not be. An attempt to introduce and study the classical abelian groups under multiset context (which we termed as multiset abelian group), normal subgroup (normal sub multiset group), and centre of the group (centre of a multiset group). In all the study results were recorded.

Keyorwds: Multiset, Multiset Group, Abelian Multiset Group, Normal sub-Multiset Group and Centre of Multiset Group

1. Introduction

Cantor is termed as the father of set theory which he propounded in 1804 and in his axioms stated that elements are not allowed to repeat in a given set, but Multiset (mset for short) allows the repetition of elements in a particular mset. It is observed from the survey of available literatures on msets and applications that the idea of mset was hinted by R. Dedikind in 1888. The mset theory which contains set theory as a special case was introduced by Cerf et al. [2]. The term mset, as noted by Knuth [4] was first suggested by N.G de Bruijn in a private communication to him. Further study was carried on by Yager [14], Blizard [1]. Other researchers ([5], [7], [8]) gave a new dimension to the multiset theory.

Msets are very useful structures arising in many areas of mathematics and computer science. Mset Topological space has been studied by Shravan and Tripathy [10]. Research on the mset theory has been gaining grounds. The research carried out so far shows a strong analogy in the behaviour of msets. It is possible to extend some of the main notion and result of sets to the setting of msets. In 2009, Girish and Sunil [15], introduced the concepts of relations, function, composition, and equivalence in msets context. Tella and Daniel ([12], [13]) have considered sets of mappings between msets and studied about symmetric groups under mset perspective. Nazmul et al. [6] improved on Tella and Daniel's work and added two axioms which marks the foundation of studying group theory in mset perspective. In this paper we present a synergy and comparison between the Nazmul et al's [6](Multigroup) and Tripathy et al's [16](Multiset Group) definition of the group theory under multiset perspective. We study the generalisation of the intersection of two multiset groups in which it is also a multiset group while that of union need not be. An attempt to study the classical normal sub group, abelian sub groups and centre of the group was studied under multiset perspective following their definition were carried out. In all these, result were recorded.

2 Preliminary definitions and notations

Definition 2.1[1]. An mset Aover the set X can be defined as a function $C_A: X \to \mathbb{N} = \{0,1,2,...\}$ where the value $C_A(x)$ denote the number of times or multiplicity or count function of x $in\, A$. For example, Let $A = [x,x,x,y,y,y,z,z]$, then $C_A(x) = 3, C_A(y) = 3, C_A(z) = 2$. $[C_A(x) = 0 \Rightarrow x \notin A]$. The mset M over the set X is said to be empty if $C_M(x) = 0\, for\, all\, x \in X$. We denote the empty mset by $\emptyset$. Then $C_\emptyset(x) = 0, \forall\, x \in X. if\; C_A(x) > 0, then\; x \in A$. We denote the set of all finite mset M over the set X to be $M(X)$. Also, elements of mset say A can belong n many times denoted as $x \in^n A$. Which means x belong to An-times.

Definition 2.2[1]: The cardinality of a mset M denoted $|M|$ or $card(M)$ is the sum of all the multiplicities of its elements given by the expression $(M) = \sum_{x \in X} c_A(x)$.

Note: Presentation of mset on paper work became a challenged as every researcher has his taught in that aspect . However the used of square brackets was adopted ([1], [9],[11]) to represent an mset and ever since then it has become a standard. For example if the multiplicity of an mset say x is 2, for y say 3,and for z say 2, it can be represented as $[x,x,y,y,y,z,z,]$, others may put it like $[x,y,z]_{2,3,2}$ or $[x^2,y^3,z^2]$ or $[x2, y3, z2]$ or $[2/x\,, 3/y\,, 2/z]$

Definition 2.3[2]: Let M be an mset drawn from a set X. The support set of M denoted byM^* is a subset of X given by $M^* = \{x \epsilon X: C_M(x) > 0\}$. that is M^* is an ordinary set. M^* is also called root or support set.

Definition 2.4[1]: Equal msets. Two msets A and B are said to be equal denoted $A = B$ if and only if for any objects $x \in X, C_A(x) = C_B(x)$. This is to say that $A = B$ if the multiplicity of every element in A is equal to its multiplicities in B and conversely. Clearly, $A = B \Longrightarrow A^* = B^*$, though the converse need not hold. For example, let $A = [a, a, b, b, c]$ and $B = [a, a, b. b, b, c, c]$ $where$ $A^* = B^* = \{a, b, c\}$ but $A \neq B$.

Definition 2.5[1]: Submsets. Let X be a set and let A and B be msets over X. A is a submset of B, denoted by $A \subseteq B$ or $B \supseteq A$, if $C_A(x) \leq C_B(x)$ $for\ all$ $x \in X$. Also if $A \subseteq B$ and $A \neq B$, then A is called proper submset of B denoted by $A \subset B$. In other words $A \subset B$ if $A \subseteq Band\ there\ exist\ at\ least\ one$ $x \in X$ such that $C_A(x) < C_B(x)$. We assert that a mset B is called the parent mset in relation to the mset A.

Definition. 2.6 [1]: Regular or Constant mset: A mset A over the set X is called regular or constant if all its elements are of the same multiplicities, i.e for any $x, y \in A$ $such\ that$ $x \neq y, C_A(x) = C_A(y)$.

Definition 2.7[1] : The notations $\wedge$ and $\vee$:[6]. The notations $\wedge$ and $\vee$ denote the minimum and maximum operator respectively for instance $C_A(x) \wedge C_A(y) = min\{C_A(x), C_A(y)\}$ and $C_A(x) \vee C_A(y) = max\{C_A(x), C_A(y)\}$.

Definition 2.8[9]: Union ($\cup$) of msets. Let A and B be two msets over a given domain set X. The union of A and B denoted by $A \cup B$is the mset defined by $C_{A\cup B}(x) = max\{C_A(x), C_B(x)\}$,

That is if object x occurrs a times in A and b times in B. Then it occurs maximum $\{a, b\}$ times in $A \cup B$, if such maximum exist.

Definition 2.9[9]: Intersection ($\cap$) of msets. Let A and B be two mset over a given domain set X. The intersection of two mset A and B denoted by $A \cap B$, is the mset for which $C_{A\cap B}(x) = min\{C_A(x), C_B(x)\}$for all $x \in X$.

In other words, $A \cap B$ is the smallest mset which is contained in both A and B. That is an objects x occuring a times in A and b in B, occurs minimum *(a, b)* times in $A \cap B$.

Definition 2.10[9]: Addition or sum of Mset. Let A and B be two msets over a given domain set X. The direct sum or arithmetic addition of A and B denoted by $A+B$ or $A \uplus B$ is the mset defined by $C_{A+B}(x) = C_A(x) + C_B(x)$ for all $x \in X$.
That is, an object x occurring a times in A and b times in B, occurs $a + b$ times in $A \uplus B$.
Thus $|A \uplus B| = |A \cup B| + |A \cap B|$.

Definition 2.11[9]: Difference of msets. Let A and B be two msets over a given domain set X. then the difference of B from A, denoted by $A - B$ is the mset such that $C_{A-B}(x) = max\{C_A(x) - C_B(x), 0\}$for all $x \in X$. If $B \subseteq A$, then $C_{A-B}(x) = C_A(x) - C_B(x)$.

It is sometimes called the arithmetic difference of B from A. If $B \nsubseteq A$ this definition still holds. It follows that the deletion of an element x from an mset A give rise to a new mset $A^{'} = A - x$ such that $C_{A'}(x) = \{C_A(x) - 1, 0\}$.

Definition 2.12[3]: Let $(m/x\, , n/y)/k$ denote an entry which means x occurs m times, y occurs n times and the ordered pair (x, y) occurs k times. Let $C_1(x, y)$ denotes the count of the first coordinate in the ordered pair (x, y) and $C_2(x, y)$ denote the count of the second coordinate in the ordered pair (x, y).

Definition 2.13[3]: Let M_1 and M_2 be two msets drawn from a set X; then the Cartesian product of M_1 and M_2 is defined as

$$M_1 \times M_2 = \{(m/x\, , n/y)/mn : x \in^m M_1, y \in^m M_2\}$$

Generally, the Cartesian product of three or more non empty msets can be gotten from the generalization of the two msets. That is the Cartesian product $M_1 \times M_2 \times \cdots \times M_n$ of non empty msets $M_1, M_2, \ldots, M_n$ is the msets of all ordered n-tupples $(m_1, m_2, \ldots, m_n)$ where $m_i \in^{ri} M_i$, $i = 1,2, \ldots, n$ and $(m_1, m_2, \ldots, m_n) \in^p M_1 \times M_2 \times \cdots \times M_n$ with $p = \prod r_i, r_i = C_{M_i}(m_i), i = 1,2, \ldots, n$. That is

$$C_{M_1 \times M_2}(m_1, m_2) = C_{M_1}(m_1). C_{M_2}(m_2)$$

For example: Let $A = [1/x\, , 2/y]$ and $B = [2/x\, , 3/z]$, then

$$A \times B = \{(1/x, 2/x)/2, (1/x, 3/z)/3, (2/y, 2/x)/4, (2/y, 3/z)/6\}$$

Theorem 2.14[3]: For any two non empty msets M_1 and M_2

$$C_{M_1 \times M_2}[(x, y)] = C_{M_1}(x).C_{M_2}(y)$$

And $|M_1 \times M_2| = |M_1|.|M_2|$. In general, $|M_1 \times M_2 \times \cdots \times M_n| = |M_1|.|M_2| \ldots |M_n|$.

Theorem 2.15 [11]. For any $M \in M(X)$, $M^* = (M^k)^* = (kM)^*$ *for any* $k \in N$ *such that* $k \geq 1$.

Theorem 2.16 [11]: Let $M, N \in M(X)$, $M \subseteq N \rightarrow M^* \subseteq N^*$

Definition 2.17[1]: The exact multiplicity axiom: $\forall x \forall y \forall n \forall m (x \in^n y \wedge x \in^m y) \rightarrow n = m$. In other words, the multiplicity with which an element belongs to a mset is unique.

Definition 2.18[1]: The axiom of extensionality: $\forall x \forall y (\forall z \forall n (z \in^n x \Leftrightarrow z \in^n y) \rightarrow x = y$. In other words, if two msets have exactly the sane elements occurring with exactly the same multiplicities, then they are equal.

Definition 2.19[6]: Let Xbe a group. An mset A over X is said to be a multigroup (mgroup for short) over X if the count function $C_A(x)$ satisfied the following conditions:

(i) $C_A(xy) \geq C_A(x) \wedge C_A(y) \forall x, y \in X$.

(ii) $C_A(x^{-1}) \geq C_A(x) \forall x \in X$

It follows immediately that:

$$C_A(x^{-1}) = C_A(x) \ \forall x \in X$$

We denote the set of all mgroups over X by $MG(X)$.

Definition 2.20[6]; Compositions.

Let $A, B \in MG(X)$, then we call $A \circ B$ as the composition between two mgroups defined as

$$C_{A \circ B}(x) = \vee\{C_A(y) \wedge C_B(z) : y, z \in X \ni yz = x\}$$

Definition 2.21[16]: Let A be a non empty mset whose maximum multiplicity is n and A^* be the root set of A. Let (m_1/x_1), $(m_2/x_2) \in A$. Then '$\circledast$' is called a binary mset composition on A if $m_1/x_1 \circledast m_2/x_2 = m_1 \circledast_1 m_2/x_1 \circledast_2 x_2$, where

(i) $' \circledast_1 '$ is a binary composition on N and $(m_1 \circledast_1 m_2) \leq n$.

(ii) $' \circledast_2 '$ is a binary composition on A^*.

Definition 2.22[16]: Let $' \circledast '$ is called a binary mset composition on A. Then A is called closed under $' \circledast '$ if $m_1/x_1 \circledast m_2/x_2 \in A$ for all $m_1/x_1, m_2/x_2 \in A$.

Definition 2.23[16]: A binary mset composition $\circledast$ on mset A is said to be associative if $m_1/x_1 \circledast (m_2/x_2 \circledast m_3/x_3) = (m_1/x_1 \circledast m_2/x_2) \circledast m_3/x_3$ for all $m_1/x_1, m_2/x_2, m_3/x_3 \in A$.

Definition 2.24[16]: A binary mset composition $\circledast$ on mset A is said to be commutative if $m_1/x_1 \circledast m_2/x_2 = m_2/x_2 \circledast m_1/x_1$, for all $m_1/x_1, m_2/x_2 \in A$.

Definition 2.25[16]: Let A be a mset with maximum multiplicity n and $' \circledast '$ be a binary mset composition on A. An element $n/e \in A$ is called the identity element of A if
$n/e \circledast m/x = m/x = m/x \circledast n/e \ for \ all \ m/x \in A.$

Definition 2.26[16]: Let A be a mset with maximum multiplicity n and $' \circledast '$ be a binary mset composition on A. An element $(m/x)^{-1} \in A$ is called the inverse element of $m/x \in A$ if
$(m/x)^{-1} \circledast m/x = n/e = m/x \circledast (m/x)^{-1} \ for \ all \ m/x \ and \ (m/x)^{-1} \in A.$

Definition 2.27[16]: Let A be a non empty mset over the set X with the binary mset composition $\circledast$. Then the pair $(A, \circledast)$ is called a multiset group of order n if the following axioms are satisfied;

(i) Closure property
i.e $(m_1/x_1 \circledast m_2/x_2) \in A, for \ all \ m_1/x_1, \ m_1/x_2 \in A.$

(ii) Associativity property

$$(m_1/x_1 \circledast m_2/x_2) \circledast m_3/x_3 = m_1/x_1 \circledast (m_2/x_2 \circledast m_3/x_3)$$

$for \ all \ m_1/x_1, \ m_1/x_2, m_3/x_3 \in A.$

(ii) Existence of identity
$n/e \circledast m/x = m/x = m/x \circledast n/e$ $for\ all\ m/x \in A.$

(iii) Existence of inverse
i.e for any $m/x \in A$ there exist an element as $r/z \in A$ called the inverse element of $m/x \in A$ if

$$m/x \circledast r/z = n/e = r/z \circledast m/x$$

Where $m_1/x_1 \circledast m_2/x_2 = (m_1 \circledast_1 m_2)/(x_1 \circledast_2 x_2)$, $\circledast_1$ and $\circledast_2$ operations on multiplicities m and x respectively such that $m_1 \circledast_1 m_2 \leq n$.

We denote the inverse element r/z of m/x by $(m/x)^{-1}$. Thus for any element $m/x \in A$, we have $m/x \circledast (m/x)^{-1} = (m/x)^{-1} \circledast m/x = n/e$

Theorem 2.28[16]: A mset A over the set X is a multiset group if and only if
$k/x \circledast (m/x)^{-1} \in A$ $for\ amy\ k/x, m/x \in A.$

Theorem 2.29[16]: In a multiset group. The identity element is unique.

Theorem 2.30[16]: For each of the element in a multiset group, there exist an unique inverse element.

Definition 2.31[16]: Let $(A, \circledast)$ be a multiset group. Then rhe multiset group is called abelian multiset group if it satisfied the commutativity property.

Definition 2.32[16]: A sub multiset group of a multiset group $(A, \circledast)$ is a sub mset of A which is a group with respect tu the same binary mset composition $' \circledast '$ as in A.

Theorem 2.33[16]: A necessary and sufficient condition for a non empty sub mset S of a multiset group $(A, \circledast)$ of order n to be a sub multiset group is that for all $m_1 / a, m_1/b \in S$ implies $(m_2/b)^{-1} \in S$.

Theorem 2.34[16]: Intersection of multiset group is again a multiset group.

Theorem 2.35[16]: The union of multiset group may not be a multiset group.

3 Main Result

Proposition 3.1: If A is a multigroup over the group X, then A is a multiset group.

Proof: Supposed that a non empty multiset A over the set X is a multigroup. Then X is a group and

(i) $C_A(xy) \geq C_A(x) \wedge C_A(y) \forall\ x, y \in X.$

(ii) $C_A(x^{-1}) \geq C_A(x) \forall x \in X$ (by definition 2.19)

Now given that e is an identity element in X, then for any $x \in A$

$$xx^{-1} = e\ which\ means$$
$$C_A(xx^{-1}) = C_A(e)$$

But $C_A(xx^{-1}) \geq C_A(x) \wedge C_A(x^{-1}) = C_A(x)$

Therefore $C_A(e) \geq C_A(x)$ in fact $C_A(e) \geq C_A(x) > 0$ $for\ some\ x$. In particular,
$C_A(e) > 0, implying\ A \neq \emptyset\ and\ e \in A.$

Now let $n = C_A(e)$ this implies that $n/e \in A$.

Also, let '$\circledast_1$' and '$\circledast_2$' be operations on the multiplicities of A and the elements of X respectively. Also let $r/z, m/x \in A$ and $\circledast$ be binary composition on A defined by $m/x \circledast r/z = (m \circledast_1 r)/(x \circledast_2 z)$(Uniqueness of multiplicity of an object in an mset and the fact that A^* is a group), then for any $m/x \in A$,
$n/e \circledast m/x = (n \circledast_1 m)/(e \circledast_2 x) = n \circledast_1 m/x = m/x$. From the operation $\circledast_2$ on A^*, $e \circledast_2 x = x$. Also for $\circledast_{"}$ $n \circledast_1 m = m$ (uniqueness axiom (Blizard, (1989))).

Hence $n/e \circledast m/x = (n \circledast_1 m)/(e \circledast_2 x) = n \circledast_1 m/x = m/x = m/x \circledast n/e$ showing the existence of identity element A.

Now let $m/x, k/y \in A$ then $m, k > 0$ and $m/x \circledast k/y = (m \circledast_1 k)/(x \circledast_2 y)$ but $x \circledast_2 y \in A^*$. In particular $x \circledast_2 y \in A$ and $(m \circledast_1 k)/(x \circledast_2 y) \in A$. Hence the closure property is satisfied.

Now since $m/x \in A$ and A^* is a subgroup of X then $x \in A^*$ and there exist an element $y \in A^*$ such that $x \circledast_2 y = y \circledast_2 x = e$ but since $y \in A^*$ then $k/y \in A$ $for\ k > 0$ then
$m/x \circledast k/y = (m \circledast_1 k)/(x \circledast_2 y) = m \circledast_1 k/e = n/e$ (uniqueness of multiplicity axiom).

Thus for any $m/x \in A$ there exist $k/y \in A$ such that $m/x \circledast k/y = n/e$.

Let $m/x\,, k/y\,,\ and\ r/z \in A$, then we show that $(m/x \circledast k/y)\ \circledast r/z\ = [(m \circledast_1 k)/(x\ \circledast_2 y)] \circledast r/z\ =$ $(m \circledast_1 k) \circledast_1 r/(x\ \circledast_2 y) \circledast_2 z$ since A^* is a subgroup $(x\ \circledast_2 y) \circledast_2 z = x\ \circledast_2 (y \circledast_2 z)$ therefore $(m \circledast_1 k) \circledast_1 r/(x\ \circledast_2 y) \circledast_2 z = m \circledast_1 (k \circledast_1 r)/x\ \circledast_2 (y \circledast_2 z)$.
Thus $(m/x \circledast k/y)\ \circledast r/z\ = m \circledast_1 (k \circledast_1 r)/x\ \circledast_2 (y \circledast_2 z)$.
However $m/x \circledast (\,k/y\ \circledast r/z) = m/x \circledast [(k \circledast_1 r)/(y\ \circledast_2 z)] = m \circledast_1 (k \circledast_1 r)/x\ \circledast_2 (y \circledast_2 z)$
Since $(x\ \circledast_2 y) \circledast_2 z = x\ \circledast_2 (y \circledast_2 z)(A^*\ is\ a\ subgroup)$ then $(m \circledast_1 k) \circledast_1 r = 8 = m \circledast_1 (k \circledast_1 r)$ uniqueness of multiplicity.
Hence the result.
The converse of this proposition need not hold, for example:
Let $X = \{1, -1, i, -i\}$ and $A = \{4/1\,, 1/-1\,, 3/i\,, 2/-i\}$.
The multigroup is not satisfied since

$$C_A(i.i) = C_A(i^2) = C_A(-1) = 1 \ngeq C_A(i) \wedge C_A(i) = \min\{3,3\} = 3$$

But the multiset group is satisfied since the axioms are satisfied on A.

Proposition 3.2 (Generalisation of the intersection of multiset group) 3.2: Let $S_1, S_2, S_3, \ldots, S_n$ be sub multiset groups of a multiset group$G$. Then their intersection$\bigcap_{i=1}^{n} S_i$ is a sub multiset group.
Proof: Let $S_1, S_2, S_3, \ldots, S_n$ be n subgroup of G. Since S_i is a subgroup.
We want to show that if $(m/x) \in S_i$ and $(n/y) \in S_i$. It implies that $(m/x)\,(n/y)^{-1} \in S_i$.
Also let $(m/x) \in S_1$, $(m/x) \in S_2$,…,$(m/x) \in S_n \Rightarrow (m/x) \in \bigcap_{i=1}^{n} S_i$
And $(n/y) \in S_1$, $(n/y) \in S_2$,…,$(n/y) \in S_n \Rightarrow (n/y) \in \bigcap_{i=1}^{n} S_i$. That is $(m/x)\,(n/y) \in \bigcap_{i=1}^{n} S_i$.
But since $(n/y) \in \bigcap_{i=1}^{n} S_i$. Then there exist $(n/y)^{-1} \in \bigcap_{i=1}^{n} S_i$ such that $(m/x)\,(n/y)^{-1} \in \bigcap_{i=1}^{n} S_i$.
Hence $\bigcap_{i=1}^{n} S_i$ is a sub multiset group.

Proposition 3.3 (Generalisation of the union of multiset group) 3.3: Let $S_1, S_2, S_3, \ldots, S_n$ be sub multiset groups of a multiset group G. Then $\bigcup_{i=1}^{n} S_i$ is a sub multiset group.
Proof: From proposition 3.2. It is clear that the union need not hold in general.

Normal Sub Multiset Group.

Definition 3.4: Let A be a mset group and let S be a sub mset group. We defined S to be a normal sub mset group if $(m/x)\,(n/y)\,(m/x)^{-1} \in S$ for any $(m/x) \in A$ and $(n/y) \in S$.
This is also called invariant sub mset group or self-conjugate subgroup.
The normal sub mset group S of A can be denoted as $S \trianglelefteq A$. Also S is said to be a normal sub mset group of A if $(m/x)S\,(m/x)^{-1} \subseteq S$ for every $(m/x) \in A$.
For example: If S is abelian then $(m/x)\,(n/y)\,(m/x)^{-1} = (n/y)$. That is

$$(m/x) \circledast (n/y) \circledast (m/x)^{-1} = (n/y)$$
$$(n/y) \circledast (m/x) \circledast (m/x)^{-1} = (n/y)$$
$$(n/y) \circledast (n/e) = (n/y)$$
$$(n \circledast_1 n/y \circledast_1 e) = (n/y)$$

That is (m/x) and (n/y) are said to be conjugate.
Two sub mset groups B and C of A are said to be conjugate if $\forall\ (m/x) \in B$ and $(n/y) \in C$ such that

$$(m/x)\,(n/y)\,(m/x)^{-1} = (n/y)$$

Definition 3.5: Abelian mset group. Let A be a mset group, then A is said to be abelian ifA^* is an abelian group and for all $(m/x), (n/y) \in A$, $(m/x)\,(n/y) = (n/y)\,(m/x)$. Abelian mset groups are also said to be commutative mset group.
For example: Let $G = \{1, -1\}$ be a group under the multiplicative operation and let $A = (2/1\,, 2/-1)$ be a m-group. Then A is abelian sinceA^* is commutative and

$$2/1 \circledast 2/-1 = 2 \circledast_1 2/1 \circledast_2 - 1 = 2/-1$$
$$2/-1 \circledast 2/1 = 2 \circledast_1 2/-1 \circledast_2 1 = 2/-1$$

Hence A is an abelian mset group.

Proposition 3.6: Every sub mset group of an abelian mset group is a normal sub mset group.
Proof: Let S be a sub m-group of a m-group A. We want to show that S is a normal sub mset group, that is $(m/x)\,(n/y)\,(m/x)^{-1} \in S$ for all $(m/x) \in A$ and $(n/y) \in S$.
Now $(m/x)\,(n/y)\,(m/x)^{-1} = (n/y)\,(m/x)\,(m/x)^{-1}$

$$= (n/y)\,(n/e), \qquad \forall\,(n/y) \in S$$

Thus $(m/x)\,(n/y)\,(m/x)^{-1} \in S$.
Hence S is a normal sub mset group of A.
Conversely, for a normal sub mset group, we want to show that it is abelian

$$(m/x)\,(n/y)\,(m/x)^{-1} = (n/y)$$

By post multiplying $\quad (m/x)\,(n/y)\,(m/x)^{-1}\,(m/x) = (n/y)\,(m/x)$

$$(m/x)\,(n/y)\,(n/e) = (n/y)\,(m/x)$$

$$(m/x)\,(n/y) = (n/y)\,(m/x)$$

Hence S is abelian sub mset group of A

Proposition 3.7: Let S be a normal sub mset group of a mset group A if and only if $(m/x)S\,(m/x)^{-1} = S$. For every $(m/x) \in A$.
Proof: Let A be a mset group and S be a sub set group of A.
We were given $(m/x)S\,(m/x)^{-1} = S\ \forall\,(m/x) \in A$. We want to show that S is a sub mset group of A.
Now $(m/x)S\,(m/x)^{-1} = S$ this implies that$(m/x)S\,(m/x)^{-1} \subseteq S$, since S is a sub mset group of A.
Conversely, let S be a sub mset group of A. We want to show that
$(m/x)S\,(m/x)^{-1} = S\ \ \forall\,(m/x) \in A$.
Since S is a normal sub mset group of A, then $(m/x)S\,(m/x)^{-1} \subseteq S \ldots (i)\ \ \forall\,(m/x) \in A$.
Now $\forall\,(m/x) \in A$, there exist $(m/x)^{-1} \in A$. Since S is a normal sub mset group of A.
$(m/x)^{-1}S[(m/x)^{-1}]^{-1} \subseteq S$ by definition $(m/x)^{-1}S\,(m/x) \subseteq S$
By pre multiplying $(m/x)\,(m/x)^{-1}S\,(m/x) \subseteq (m/x)S$

$$(n/e)S\,(m/x) \subseteq (m/x)S$$

$$S\,(m/x) \subseteq (m/x)S$$

By post multiplying

$$S\,(m/x)\,(m/x)^{-1} \subseteq (m/x)S\,(m/x)^{-1}$$

And

$$S\,(n/e) \subseteq (m/x)S\,(m/x)^{-1}$$

$$S \subseteq (m/x)S\,(m/x)^{-1} \ldots (ii)$$

From (i) and (ii) we have $(m/x)S\,(m/x)^{-1} = S$.
Hence the result.

Centre of Multiset Group.
Definition 3.8: Let A be a mset group over a group X. We defined the centre of A denoted as $Z(A) = \{\mathit{for\ all}\ (n/y) \in A, (m/x)\,(n/y) = (n/y)\,(m/x),\ (m/x) \in A\}$. Thus $Z(A)$ is also said to be a normal sub mset group of A.
For example: Let $X = \{1, -1\}$ and let $A = \{3/1\,, 2/-1\,\}$, then $Z(A) = 3/1 \circledast 2/1 = 3/-1 = 2/-1 \circledast 3/1$.

Definition 3.9 (Commutator of mset group) 3.9: Let A be a mset group over X for all $x, y \in X$, we defined the commutator of A over X as $(m/x)^{-1}\,(n/y)^{-1}\,(m/x)\,(n/y)$, it is denoted by $[(m/x), (n/y)]$.
Remark:
- (i) The mset commutator of mset group is a sub mset group over the said group.
- (ii) The sub mset group generated by the set of all mset commutators is called the commutators of mset group.

Proposition 3.10: Let A be a mset group over a group X, then the inverse of a commutator is a commutator.
Proof: Let $[(m/x), (n/y)]$ be a commutator, that is $[(m/x), (n/y)] = (m/x)^{-1}\,(n/y)^{-1}\,(m/x)\,(n/y)$. We want to show that $[(m/x), (n/y)]^{-1}$ is a commutator.
Now
$[(m/x), (n/y)]^{-1} = ((m/x)^{-1})^{-1}((n/y)^{-1})^{-1}((m/x))^{-1}((n/y))^{-1} = (m/x)\,(n/y)\,(m/x)^{-1}\,(n/y)^{-1} = (m/x)^{-1}\,(n/y)^{-1}\,(m/x)\,(n/y)$.
Thus $[(m/x), (n/y)]^{-1}$ is a commutator.

Proposition 3.11: Let A be a mset group over a group X. If A is an abelian mset group, then the set of all mset commutators equals $\{1\}$.
Proof: Suppose $x, y \in X$ and $(m/x)\,(n/y) \in A$ and if A is abelian, then
$(m/x)\,(n/y) = (n/y)\,(m/x)$ and
$[(m/x), (n/y)] = (m/x)^{-1}\,(n/y)^{-1}\,(m/x)\,(n/y) = \{1\}$. But the set of all commutators of A is the sub mset group of A and generated by $\{1\}$. Then the set of all commutators of $A = \{1\}$.

Thus $[(m/x),(n/y)] = (m/x)^{-1}\,(n/y)^{-1}\,(m/x)\,(n/y) = \{1\}$. This implies that

$$(m/x).(m/x)^{-1}\,(n/y)^{-1}\,(m/x)\,(n/y) = (m/x).1$$
$$(n/y).(n/y)^{-1}\,(m/x)\,(n/y) = (n/y).(m/x)$$
$$(m/x)\,(n/y) = (n/y).(m/x)$$

Thus A is abelian and the set of all mset commutators equals $\{1\}$.

Proposition 3.12: Let A be a mset group over a group X. If A is any mset group, then $Z(A)$ is a normal sub mset group.

Proof: Let $Z(A) \neq \emptyset$, then $(n/e) \in Z(A)$ since

$$(m/x)\,(n/e) = (n/e).(m/x)$$

Where e is the identity element with multiplicity n, for all $(m/x) \in Z(A)$.

Now given $m_1/x_1,\ m_1/x_2\ \in Z(A)$. Then $(m/x)[(m_1/x_1).(m_2/x_2)^{-1}] = [(m/x).(m_1/x_1)](m_2/x_2)^{-1} = (m_1/x_1)[(m/x)(m_2/x_2)^{-1}] = [(m_1/x_1).(m_2/x_2)^{-1}]\,(m/x)$.

Since $(m/x)(m_2/x_2) = (m_2/x_2)\,(m/x)$, it implies $(m_2/x_2)^{-1}\,(m/x) = (m/x)(m_2/x_2)^{-1}$. It follows that $Z(A)$ is a sub mset group of A.

Also if $(m/x) \in A$ and $(m_1/x_1) \in Z(A)$,

$$(m/x)(m_1/x_1) = (m_1/x_1)\,(m/x)$$

And so $(m_1/x_1) = (m/x)^{-1}(m_1/x_1)\,(m/x)$

Hence $(m/x)^{-1}(m_1/x_1)\,(m/x) \in Z(A)\ for\ all\ \ (m/x) \in A\ and\ (m_1/x_1) \in Z(A)$.

Thus $Z(A)$ is a normal sub mset group of A.

Proposition 3.13: Let M be a mset group over a group X. Let A be a sub mset group of M and B a normal sub mset group of M. Then AB is a sub mset group of M.

We defined $AB = \{z/z = (m/x)\,(n/y),\ \forall\ (m/x) \in A\ and\ \ (n/y) \in B\ \}$.

Proof: Let $AB \neq \emptyset$, then $(n/e) \in AB$, where e is the identity element with multiplicity n, that is $(n/e)\,(n/e) \in AB\ for\ \ (n/e) \in A\ and\ \ (n/e) \in B$.

Now let $z_1, z_2 \in AB$ such that $z_1 = (m_1/x_1)(n_1/y_1), z_2 = (m_2/x_2)(n_2/y_2)$ where $(m_i/x_i) \in A$ and $(n_i/y_i) \in B$.

Then $z_1{z_2}^{-1} = (m_1/x_1)(n_1/y_1)(m_2/x_2)^{-1}(n_2/y_2)^{-1}$

$$= (m_1/x_1)(n_1/y_1)(n_2/y_2)^{-1}(m_2/x_2)^{-1}$$
$$= (m_1/x_1)(m_2/x_2)^{-1}((m_2/x_2)^{-1})^{-1}(n_3/y_3)(m_2/x_2)^{-1}$$
$$= [(m_1/x_1)(m_2/x_2)^{-1}][(m_2/x_2)(n_3/y_3)(m_2/x_2)^{-1}]$$

Thus $[(m_1/x_1)(m_2/x_2)^{-1}] \in A$ is a sub mset group of M.

$[(m_2/x_2)(n_3/y_3)(m_2/x_2)^{-1}] \in B$ a normal sub mset of M.

Hence $z_1{z_2}^{-1} \in AB$ is a sub mset group of M.

Proposition 3.14: Let M be a mset group over a group X. Let A and B be normal sub mset groups of M. Then $A \cap B$ is a sub mset group of M.

Proof: Since A and B are normal sub mset groups of M, then they are sub mset groups of M, that is for all $(m/x),(n/y) \in A$ and $(m/x)\,(n/y)^{-1} \in A$, also same reason for B. That is $(m/x)\,(n/y)^{-1} \in A \cap B$. We want to show that $A \cap B$ is normal in M.

Let $(n/y) \in A \cap B$ and let $(m/x) \in M$. Since A is a normal sub mset group of M. Then $(m/x)\,(n/y)\,(m/x)^{-1} \in A$ as (n/y) is in A, which implies A is normal sub mset group of M. Also $(m/x)\,(n/y)\,(m/x)^{-1} \in B$ as (n/y) is in $B, (m/x) \in M$.

Thus $(m/x)\,(n/y)\,(m/x)^{-1} \in A \cap B$.

Definition (Centralizer) 3.15: Let M be a mset group over a group X and let A be a submset group of M. We defined the centralizer denoted as $C(A)$ of A in M, by

$C(A) = \{(m/x) \mid (m/x) \in M\ \ and\ \ (m/x)\,(n/y) = (n/y)\,(m/x), (n/y) \in A\}$.

Definition (Normalizer) 3.16: Let M be a mset group over a group X and let A be a submset group of M. We defined the normalizer denoted as $N(A)$ of A in M, by

$N(A) = \{(n/y)/\,(n/y) \in M\ \ and\ A\,(n/y) = (n/y)A\ or\ \ (n/y)A\,(n/y)^{-1} = A\ \}$.

Proposition 3.17: Let A be a mset group over a group X. Then $Z(A)$ is a sub mset group of A.

Proof: Suppose $A \neq \emptyset$, then let $(n/e) \in A$ and

$(m/x)\,(n/e) = (n/e)\,(m/x) = (m/x),\ \forall\ \ (m/x) \in A$.

Then $(n/e) \in Z(A)$ and implies that $Z(A) \neq \emptyset$.
Now let $(p/r), (q/z) \in Z(A)$, then let $(m/x)\,(p/r) = (p/r)\,(m/x)$ and
$(m/x)\,(q/z) = (q/z)\,(m/x)$.
We want to show that $(p/r)\,(q/z)^{-1} \in Z(A)$.
Since $(m/x)\,(q/z) = (q/z)\,(m/x)$, this implies that

$$(q/z)^{-1}[(m/x)\,(p/r)]\,(q/z)^{-1} = (q/z)^{-1}[(p/r)\,(q/z)]\,(q/z)^{-1}$$
$$(m/x)\,(q/z)^{-1} = (q/z)^{-1}\,(m/x)\ \forall\ (m/x) \in A$$

To show $(p/r)\,(q/z)^{-1}$ that is $(p/r)\,(q/z)^{-1}\,(m/x) = (m/x)\,(p/r)\,(q/z)^{-1}$

$$\forall\ (m/x) \in A$$
$$(p/r)[(q/z)^{-1}\,(m/x)] = (p/r)[(m/x)\,(q/z)^{-1}]$$
$$= [(p/r)\,(m/x)]\,(q/z)^{-1}\,(m/x)$$

$\Rightarrow (p/r)\,(q/z)^{-1}\,(m/x) = (m/x)\,(p/r)\,(q/z)^{-1}$
Thus $(p/r)\,(q/z)^{-1} \in Z(A)$.

Proposition 3.18: The normalizer of a submset group M of a mset group A is a sub mset group of M.
Proof: Let $(n/e) \in M$, where e is the identity element. This implies that
$(n/e)A = A\,(n/e) = A$ which mean $(n/e) \in N(A)$ and so $N(A) \neq \emptyset$.
Now let $(m/x), (n/y) \in N(A)$, this implies that
$(m/x)A\,(m/x)^{-1} = A$ and $(n/y)A\,(n/y)^{-1} = A$, also if
$A = (n/y)A\,(n/y)^{-1}$ then

$$(m/x)\,(n/y)^{-1}A[(m/x)\,(n/y)^{-1}]^{-1} = (m/x)\,(n/y)^{-1}A\,(n/y)\,(m/x)^{-1}$$
$$= (m/x)A\,(m/x)^{-1} = A$$

This implies that $(m/x)\,(n/y)^{-1} \in N(A)$.
Thus $N(A)$ is a sub mset group of M.

Proposition 3.19: Let A and B be normal sub mset groups of a mset group M over the group X. If $AB = \{(m/x)\,(n/y) : (m/x) \in A, (n/y) \in B\}$, then AB is a sub mset group of M and abelian.
Proof: Suppose $\forall\ (p/r) \in M$ we say $(p/r) = (p/r)\,(m/x)\,(p/r)^{-1}$.
Also $\forall\ (q/z) \in M$ we say $(q/z) = (q/z)\,(n/y)\,(q/z)^{-1}$.
Now we want to show that AB is a sub mset group of M. If $(p/r)\,(q/z)^{-1} \in M$, then

$$(p/r)\,(q/z)^{-1} = (p/r)\,(m/x)\,(p/r)^{-1}[(q/z)\,(n/y)\,(q/z)^{-1}]^{-1}$$
$$= (p/r)\,(m/x)\,(p/r)^{-1}\,(q/z)^{-1}\,(n/y)^{-1}((q/z)^{-1})^{-1}$$
$$= (p/r)\,(m/x)\,(p/r)^{-1}\,(q/z)^{-1}\,(n/y)^{-1}\,(q/z)$$
$$= (p/r)\,(q/z)\,(m/x)\,(n/y)^{-1}\,[(p/r)\,(q/z)]^{-1}$$

Thus $(m/x)\,(n/y)^{-1} = (p/r)\,(q/z)^{-1} \in M$ showing is a sub mset group.
Also $(m/x)\,(n/y) = (p/r)\,(q/z)\,(m/x)\,(n/y)\,(p/r)^{-1}\,(q/z)^{-1}$

$$= (p/r)\,(q/z)\,(m/x)\,(n/y)\,[(q/z)(p/r)]^{-1}$$
$$= (q/z)\,(p/r)\,(n/y)\,(m/x)[(p/r)\,(q/z)]^{-1}$$
$$= (n/y)\,(m/x)$$

Thus $(m/x)\,(n/y) = (n/y)\,(m/x)$ is abelian.

Proposition 3.20: The conjugacy relation of a sub mset group A of a mset group M is an equivalence relation.
Proof: Let the conjugacy relation be denoted $\sim$.
Now $\forall\ (m/x), (n/y) \in A$, if (m/x) and (n/y) are conjugates then $\forall\ (q/z) \in M$, then $(m/x) = (q/z)\,(m/x)\,(q/z)^{-1}$ that is $(m/x) \sim (q/z)\,(m/x)\,(q/z)^{-1}$.
Thus the relation is reflexive.
Again, $(m/x) = (q/z)\,(n/y)\,(q/z)^{-1}$ that is $(m/x) \sim (q/z)\,(n/y)\,(q/z)^{-1}$.
Thus the relation is symmetric.
Also, $\forall\ (m/x), (n/y)\ and\ (p/r) \in A$ as conjugates point elements and $\forall\ (q/z) \in M$.
Then $(m/x) = (q/z)\,(n/y)\,(q/z)^{-1}$ that is $(m/x) \sim (q/z)\,(n/y)\,(q/z)^{-1}$ and
$(n/y) = (q/z)\,(p/r)\,(q/z)^{-1}$ which implies that $(n/y) \sim (q/z)\,(p/r)\,(q/z)^{-1}$. Thus
$(m/x) = (q/z)[(q/z)\,(p/r)\,(q/z)^{-1}]\,(q/z)^{-1} = (q/z)\,(p/r)\,(q/z)^{-1}$ that is
$(m/x) \sim (q/z)\,(p/r)\,(q/z)^{-1}$ is transitive.
Hence $\sim$ is an equivalence relation.

4 Conclusions

Tella and Daniel began the study of 'multigroup' and Nazmul et al build on what they have done. Our study is a build-up from the perspective of Tripathy et al([16]) (on multiset group) which they termed as'multiset group'. The foundation they laid gave an insight for the study of abelian multiset group. In the study we consider, introduced and study the normal sub multiset group, the centre of a multiset group, and the commutator of a multiset group. We first of all established the synergy and contrast between the definition of Nazmul et al, (2013) and that of Tripathy et al, (2018). On 'Multigroup' and 'Multiset group' respectively. Where we show that every multigroup is a multiset group but the converse need not true. Other aspects of group theory such as the conjugacy and the normal sub multiset group, the centralizer and the normalizer of nultiset group from the perspective of Tripathy, et al were studied and in all results were recorded.

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