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# Ricci Solitons on $(\varepsilon)$-Para Sasakian Manifolds Admitting Concircular Curvature Tensor

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**Abstract:** The object of the present paper is to study Ricci solitons in $(\varepsilon)$-para Sasakian manifolds satisfying $S(\xi,X).\bar{C}=0$, $R(\xi,X).\bar{C}=0$, $\bar{C}(\xi,X).S=0$, $\bar{C}(\xi,X).R=0$, $R(\xi,X).\bar{C}=0$, where $\bar{C}$ is concircular curvature tensor.
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## 1. Introduction:

In the differential geometry, the Ricci flow is an intrinsic geometric flow, which was introduced by R. Hamilton ([11], [12]).The Ricci flow is a process that deforms the metric of a Riemannian manifold in a way formally analogous to the diffusion of heat, smoothing our irregularities in the metric. The Ricci flow equation is the evolution equation

$$\frac{d}{dt} g_{ij}(t) = -2\, R_{ij}$$

for a Riemannian metric $g_{ij}$, where $R_{ij}$ is the Ricci curvature tensor. Hamilton showed that there is a unique solution to this equation for an arbitrary smooth metric $g_{ij}$. Hamilton on a closed manifold over a sufficient short time. He also showed that Ricci flow preserves positivity of Ricci curvature tensor in three dimensions and the curvature operator in all dimensions. Ricci solitons are Ricci flows that may change their size but not their shape up to diffeomorphisms.

A significant 2-dimensional example of Ricci soliton is the cigar solution which is given by the metric $(dx^2 + dy^2)/(e^{4t} + x^2 + y^2)$ on the Euclidean plane. Although this metric shrinks under the Ricci flow, its geometry remains the same. Such a solutions are called steady Ricci solitons.

A Ricci soliton is a triple $(g, v, \lambda)$ with $g$ a Riemannian metric, v a vector field and $\lambda$ a real scalar such that

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \qquad (1.1)$$

where S is a Ricci tensor of $M^n$ and $\mathcal{L}_V$ denote the Lie –derivative along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding accordingly as real scalar $\lambda$ is negative, zero and positive respectively.Ricci solitons were studied by several authors in contact and Lorentzian manifold, Para Sasakian manifold such as Sharma [20], Bagewadi and Ingalahalli [1], Nagaraja and Premalatha [15], Bagewadi [1], Pandey, Patel and Singh [17] et all and others.

On the other hand, the study of manifolds with indefinite metrics is of interest from the stand point of physics and relativity. Manifolds with indefinite metrics have been studied by several authors. In 1993, Bejancu and Duggal [3] introduced the concept of $(\varepsilon)$-Sasakian manifolds and Xufeng and Xiaoli [22] established that these manifolds are real hyper- surfaces of indefinite Kahlerian manifolds. De and Sarkar [7] introduced $(\varepsilon)$-para Sasakian manifolds and studied some curvature conditions on it. Singh, Pandey, Pandey and Tiwari [16], Patel, Pandey and Singh {[18],[19]}, established the relation between semi-symmetric metric connection and Riemannian connection on $(\varepsilon)$-para Sasakian manifolds and have studied several curvature conditions.

Motivated by these studies, we study Ricci solitons in $(\varepsilon)$-para Sasakian manifolds. In this paper, we have studied Ricci solitons in $(\varepsilon)$-para Sasakian manifolds satisfying $S(\xi,X).\bar{C} = 0$, $R(\xi, X).\bar{C} =0$, $\bar{C}(\xi, X).S=0$, $\bar{C}(\xi, X).R =0$, $R(\xi, X).\bar{C} =0$, where $\bar{C}$ is concircular curvature tensor of the manifold.

## 2. $(\varepsilon)$-Para Sasakian manifolds:

Let $M^n$ be an almost paracontact manifold equipped with an almost paracontact structure $(\phi, \xi, \eta)$ consisting of a tensor field $\phi$ of type (1,1), a vector field $\xi$ and a one $\eta$ satisfying

$$\phi^2 X = X - \eta(X)\xi, \qquad (2.1)$$

$$\eta(\xi) = 1, \qquad (2.2)$$

$$\phi(\xi)=0, \qquad (2.3)$$

and

$$\eta o \phi = 0. \qquad (2.4)$$

Let $M^n$ be an n-dimensional almost paracontact manifold and $g$ be semi-Riemannian metric with index ($g$)=v such that

$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \tag{2.5}$$

where $\varepsilon = \pm 1$. In this case, $M^n$ is called an ($\varepsilon$)-almost paracontact metric manifold equipped with an ($\varepsilon$)-almost paracontact structure $(\phi, \xi, \eta, g)$,[18]. In particular, if index ($g$)=1, then an ($\varepsilon$)-almost paracontact metric manifold will be called a Lorentzian almost paracontact manifold. If in case, the metric is positive definite, then an ($\varepsilon$)-almost paracontact metric manifold is the almost paracontact metric manifold.

In view of equations (4), (5) and (7), we have

$$g(\phi X, Y) = g(X, \phi Y) \tag{2.6}$$

and

$$\varepsilon g(X, \xi) = \eta(X), \tag{2.7}$$

for every $X, Y \epsilon TM^n$. From equation (9), it follows that

$$\varepsilon = g(\xi, \xi), \tag{2.8}$$

i.e. the structure of vector field $\xi$ is never light-like.An ($\varepsilon$)-almost paracontact metric manifold (resp., a Lorentzian almost paracontact manifold $(M^n, \phi, \xi, g, \varepsilon)$[18], is said to space-like ($\varepsilon$)-almost paracontact metric manifold (respectverly a space-like Lorentzian almost paracontact manifold), if $\varepsilon = 1$ and $M^n$ is said to be a time–like ($\varepsilon$)-almost paracontact manifold (respectively a Lorentzain almost paracontact manifold ) if $\varepsilon = -1$.

An ($\varepsilon$)-almost paracontact metric structure is called an ($\varepsilon$)-Para Sasakian structure if

$$(\nabla_X \phi)(Y) = -g(X, \phi Y)\xi - \epsilon\eta(Y)\phi^2 X, \qquad X, Y \epsilon TM^n, \tag{2.9}$$

where $\nabla$ is the Levi-Civita connection. A manifold $M^n$ endowed with an ($\varepsilon$)–para Sasakian structure is called an ($\varepsilon$)-para Sasakian manifold. For $\varepsilon = 1$ and $g$ Riemannian metric, $M^n$ is the usual para Sasakian manifold [18]. For $\varepsilon$=-1, $g$ Larentzian metric and $\xi$ replaced by $-\xi$, $M^n$ becomes a Lorentzian para Sasakian manifold. In an ($\varepsilon$)-para Sasakian manifold, we have

$$\nabla_X \xi = \varepsilon\phi X, \tag{2.10}$$

$$\Omega(X, Y) = \varepsilon g(\phi X, Y) = (\nabla_X \eta)(Y), \tag{2.11}$$

for every $X, Y \epsilon TM^n$, where $\Omega$ is the fundamental 2-form. In an ($\varepsilon$)-almost para Sasakian manifold $M^n$, the following relations holds.

$$R(\xi, X)Y = -\varepsilon g(X, Y)\xi + \varepsilon\eta(Y)X, \tag{2.12}$$

$$R(X, Y)\xi = -\varepsilon\eta(Y)X + \varepsilon\eta(X)Y. \tag{2.13}$$

In an n-dimensional ($\varepsilon$)-para Sasakian manifold $M^n$, the Ricci tensor S satisfies

$$\eta(R(X, Y)Z) = \varepsilon[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \tag{2.14}$$

Let ( $g$, V, $\lambda$ ) be a Ricci solitons in an ($\varepsilon$)-Kenmotsu manifold. From equation (2.9), we have

$$(L_\xi g)(X, Y) = -2[\varepsilon g(X, Y) - \eta(X)\eta(Y)]. \tag{2.15}$$

In view of equations (1.1) and (2.15), we have

$$S(X, Y) = [\varepsilon g(\phi X, Y) - \lambda g(X, Y)], \tag{2.16}$$

yields that

$$S(X, \xi) = -\varepsilon\lambda\,\eta(X), \tag{2.17}$$

$$QX = \varepsilon\phi X - \lambda X, \tag{2.18}$$

$$r = n\,[\phi X - \lambda X]\,. \tag{2.19}$$

The concircular curvature tensor $\bar{C}$ is defined as [14]

$$\bar{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y]. \tag{2.20}$$

By virtue of equations (2.7) and (2.12), and using equation (2.20) as X=$\xi$, the concircular curvature tensor on ($\epsilon$)-Para Sasakian manifold takes the form

$$\bar{C}(\xi, Y)Z = -[\varepsilon + \frac{r}{n(n-1)}]g(Y, Z)\xi + [\varepsilon + \frac{r\varepsilon}{n(n-1)}]\eta(Z)Y], \tag{2.21}$$

Again on putting Z=$\xi$ , in equation (2.20) and by the use of equation (2.2), (2.3), (2.7) and (2.13), we obtain

$$\bar{C}(X, Y)\xi = -[1 + \frac{r}{n(n-1)}][\varepsilon\eta(Y)X - \varepsilon\eta(X)Y]. \tag{2.23}$$

which gives

$$\eta(\bar{C}(X, Y)Z) = [\varepsilon - \frac{r}{n(n-1)}][g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]. \tag{2.24}$$

**Example:** Let $R^3$ be the 3-dimensional real number space with a co-ordinate system (X,Y,Z) we define

$$\eta = dz - Ydx, \quad \xi = \frac{\partial}{\partial Z}, \quad \phi\left(\frac{d}{dx}\right) = -\frac{d}{dx} - y\frac{d}{dz}, \quad \phi\left(\frac{d}{dy}\right) = -\frac{d}{dy}, \quad \phi\left(\frac{d}{dz}\right) = 0,$$

$g_1=(dx)^2+dy^2-\eta\otimes\eta,$
$g_2=(dx)^2+(dy)^2+(dz)^2-y(dx\otimes dz+dz\otimes dx),$
$g_3=-(dx)^2+(dy)^2+(dz)^2-y(dx\otimes dz+dz\otimes dx).$

Then the $(\phi,\xi,\eta)$ is an almost paracontact structure in $R^3$. The set the $(\phi,\xi,\eta,g_1)$ is a time –like Lorentzain paracontact structure. Moreover, trajectories of the time-like structure vector $\xi$ are geodesics. The set $(\phi,\xi,\eta,g_2)$ is space-like Lorentzian almost paracontact structure. The set the $(\phi,\xi,\eta,g_3)$ is a space-like $(\varepsilon)$-almost paracontact structure the $(\phi,\xi,\eta,g_3,\varepsilon)$ with index $(g_3)$=2.

## 3. Ricci Solitons in (ε)-Para Sasakian Manifolds Satisfying $S(\xi,X).\bar{C}=0$

Using the following equations

$$\begin{aligned}S(X,\xi).\bar{C})(Y,Z)U&=((X\wedge_S\xi).\bar{C})(Y,Z)U\\&=(X\wedge_s\xi)\bar{C}(Y,Z)U+\bar{C}((X\wedge_s\xi)(Y,Z)U)\\&+\bar{C}(X,(X\wedge_s\xi)Y)U)+\bar{C}(Y,Z)(X\wedge_s\xi)U,\end{aligned}\tag{3.1}$$

where the endomorphism $(X\wedge_S Y)$ is defined by

$$(X\wedge_S Y)Z=S(Y,Z)X-S(X,Z)Y.\tag{3.2}$$

Now, from equations (3.1) and (3.2), we have

$$\begin{aligned}(S(X,\xi).\bar{C})(Y,Z)U&=S(\xi,\bar{C}(Y,Z)U)X-S(X,\bar{C}(Y,Z)U)\xi+S(\xi,Y)\bar{C}(X,Z)U)\\&-S(X,Y)\bar{C}(\xi,Z)U+S(\xi,Z)\bar{C}(Y,X)U-S(X,Z)\bar{C}(Y,\xi)U\\&+S(\xi,U)\bar{C}(Y,Z)X-S(X,U)\bar{C}(Y,Z)\xi.\end{aligned}\tag{3.3}$$

Assuming $(S(X,\xi).\bar{C})(Y,Z)U=0$, then above equation reduces to

$$\begin{aligned}&S(\xi,\bar{C}(Y,Z)U)X-S(X,\bar{C}(Y,Z)U)\xi+S(\xi,Y)\bar{C}(X,Z)U-S(X,Y)\bar{C}(\xi,Z)U\\&+S(\xi,Z)\bar{C}(Y,X)U-S(X,Z)\bar{C}(Y,\xi)U+S(\xi,U)\bar{C}(Y,Z)X\\&-S(X,U)\bar{C}(Y,Z)\xi=0.\end{aligned}\tag{3.4}$$

Taking the inner product of above equation with ξ and using equation (2.3), (2.7),we get

$$\begin{aligned}&\varepsilon\eta(X)S(\xi,\bar{C}(Y,Z)U)-S(X,\bar{C}(Y,Z)U)+\varepsilon S(\xi,Y)\eta(\bar{C}(X,Z)U)-\varepsilon S(X,Y)\eta(\bar{C}(\xi,Z)U)\\&+\varepsilon S(\xi,Z)\eta(\bar{C}(Y,X)U)-\varepsilon S(X,Z)\eta(\bar{C}(Y,\xi)U)\\&+\varepsilon S(\xi,U)\eta(\bar{C}(Y,Z)X)-\varepsilon S(X,U)\eta(\bar{C}(Y,Z)\xi)=0.\end{aligned}\tag{3.5}$$

In virtue of above equations (2.17) and (2.23), we get

$$\begin{aligned}S(X,\bar{C}(Y,Z)U)&=\left[\varepsilon-\frac{r}{n(n-1)}\right][-\varepsilon S(X,Y)g(Z,U)+S(X,Y)\eta(U)\eta(Z)\\&-S(X,Z)\eta(U)\eta(Y)+\varepsilon S(X,Z)g(Y,U)\\&-\lambda g(Z,X)\eta(U)\eta(Y)+\lambda g(Y,X)\eta(U)\eta(Z)],\end{aligned}\tag{3.6}$$

which by virtue of equation (2.5) and (2.17) in Putting $Y=\xi$, gives

$$\begin{aligned}&\left[\varepsilon-\frac{r}{n(n-1)}\right]\lambda[(1-\varepsilon)\{g(Z,U)\eta(Y)-g(Y,U)\eta(Z)\}\\&-2\varepsilon\eta(Y)\eta(U)\eta(Z)]=0.\end{aligned}\tag{3.7}$$

Putting $Z=U=e_i$ and taking summation over i, $1\le i\le n$, we get

$$\left[\varepsilon-\frac{r}{n(n-1)}\right]2\lambda(1+\varepsilon)\eta(Y)=0.\tag{3.8}$$

In virtue of equations (2.5), (2.2) and (2.3) Putting $Y=\xi$, we get

$$\left[\varepsilon-\frac{r}{n(n-1)}\right]2\lambda(1+\varepsilon)=0.\tag{3.9}$$

$$\lambda=0 \text{ or } \left[\varepsilon-\frac{r}{n(n-1)}\right]=0,$$

which shows that λ is steady .Thus we can state as follows -

**Theorem (3.1):** Ricci Soliton in $(\varepsilon)$-para Sasakian manifolds with ξ as space–like vector field satisfying $S(\xi,X).\bar{C}=0$, λ is steady.

Now, suppose ξ is space –like vector field in $(\varepsilon)$-para Sasakian manifolds, then from equation (3.8), we obtain

$$\lambda=0 \text{ or } \lambda>0,$$

which shows that λ is either steady or shrinking. Thus we can state as follows -

**Theorem (3.2):** Ricci Soliton in $(\varepsilon)$-para Sasakian manifolds with ξ as space–like vector field satisfying $S(\xi,X).\bar{C}=0$, is either steady or shrinking.

## 4. Ricci Soliton in (ε)-Para Sasakian Satisfying $R(\xi,X).\bar{C}=0$.

Let us suppose $R(\xi,X).\bar{C}=0$, that is

$$(R(\xi,X).\bar{C})(Y,Z)U=0,$$

which gives

$$R(\xi,X)\bar{C}(Y,Z)U-\bar{C}(R(\xi,X)Y,Z)U-\bar{C}(Y,R(\xi,X)Z)U-\bar{C}(Y,Z)R(\xi,X)U=0.\tag{4.1}$$

In view of equation (2.14), above equation reduces to

$$\varepsilon\eta(\bar{C}(Y, Z)U)X - \varepsilon g(X, \bar{C}(Y, Z)U)\xi + \varepsilon g(X, Y)\bar{C}(\xi, Z)U + \varepsilon\eta(Y)\bar{C}(X, Z)U + \varepsilon g(X, Z)\bar{C}(Y, \xi)U - \varepsilon\eta(Z)\bar{C}(Y, X)U + \varepsilon g(X, U)\bar{C}(Y, Z)\xi - \varepsilon\eta(U)\bar{C}(Y, Z)X = 0. \quad (4.2)$$

Now, taking the inner product of above equation with ξ, and using equation (2.2) and (2.3), we get

$$\eta(X)\eta(\bar{C}(Y, Z)U) - \varepsilon g(X, \bar{C}(Y,Z)U) + g(X, Y)\eta(\bar{C}(\xi, Z)U) + \eta(Y)\eta(\bar{C}(X, Z)U) + g(X, Z)\eta(\bar{C}(Y, \xi)U) - \eta(Z)\eta(\bar{C}(Y, X)U) + g(X, U)\eta(\bar{C}(Y, Z)\xi) - \eta(U)\eta(\bar{C}(Y, Z)X) = 0. \quad (4.3)$$

By virtue of equation (2.24), above equation takes the form

$$\varepsilon g(X, \bar{C}(Y, Z)U) = K_2[-(\varepsilon - 1)\{-g(X, Y)\eta(Z) + g(X, Z)\eta(Y)\}\eta(U) + 2g(Z, U)\eta(X)\eta(Y) - g(Y, U)\eta(X)\eta(Z) - g(X, U)\eta(Y)\eta(Z) - g(X, Z)g(Y, U) - g(X, Y)g(Z, U)]. \quad (4.4)$$

which on putting $X=\xi$, we get

$$K_2[g(Z, U)\eta(Y) - \varepsilon\eta(U)\eta(Y)\eta(Z) + (\varepsilon - 1)g(Z, U)\eta(Y) - \varepsilon g(Y, U)\eta(Z)] = 0, \quad (4.5)$$

Putting $Y=Z=\xi$ using equation, we get

$$K_2[3\varepsilon\eta(U)] = 0, \quad \text{where } K_2=[\varepsilon - \frac{r}{n(n-1)}] \quad (4.6)$$

In view of equation $U=\xi$, we get

$$\varepsilon[\varepsilon - \frac{r}{n(n-1)}]=0, \quad \text{where } r=n(\phi X - \lambda X) \quad (4.7)$$

$$\varepsilon[\varepsilon - \frac{(\phi\xi - \lambda\xi)}{(n-1)}]=0,$$

$$\lambda < 0,$$

which shows that λ is shrinking. Thus we can state as follows -

**Theorem (4.1):** Ricci Soliton in $(\varepsilon)$-para Sasakian manifolds with ξ as space vector field satisfying $R(\xi, X).\bar{C} = 0$, is shrinking .

## 5. Ricci Soliton in $(\varepsilon)$-Para Sasakian Manifold Satisfying $\bar{C}(\xi, X).S=0$.

**Let us suppose $\bar{C}(\xi, X).S=0$, gives**

$$S(\bar{C}(\xi, X)Y,Z)+S(Y,\bar{C}(\xi, X)Z)=0. \quad (5.1)$$

By virtue of equation (2.21) above equation takes the form

$$-[\varepsilon + \frac{r}{n(n-1)}][g(X, Y)S(\xi, Z) + g(X, Z)S(Y, \xi)] + 2[\varepsilon + \frac{r\varepsilon}{n(n-1)}]\eta(Z)S(X, Y) = 0. \quad (5.2)$$

In view of equations (2.17) and (2.18), above equation takes the form

$$[\varepsilon + \frac{r}{n(n-1)}][\lambda\varepsilon g(X, Y)\eta(Z) - \lambda\varepsilon g(X, Z)\eta(Y)] + 2[\varepsilon + \frac{r\varepsilon}{n(n-1)}]\eta(Z)S(X, Y) = 0. \quad (5.3)$$

Putting $X=Y=e_i$, and taking summation over i, $1\leq i \leq n$, we get

$$[\varepsilon + \frac{(\phi X - \lambda X)\varepsilon}{(n-1)}]n(\phi X - \lambda X)\eta(Z)=0, \quad (5.4)$$

Taking Inner product with $\xi$, using equation (2.2), (2.3) and (2.4), we obtain

$$\lambda = 0, \text{ or } \lambda < 0,$$

which shows that λ is steady or shrinking .Thus we can state as follows -

**Theorem (5.1):** Ricci Soliton in $(\varepsilon)$-para Sasakian manifolds with ξ as space –like vector field satisfying $\bar{C}(\xi, X).S=0$, is steady or shrinking.

## 6. Ricci Solitons in $(\varepsilon)$-Para Sasakian Satisfying $\bar{C}(\xi, X).R=0$.

Let $\bar{C}(\xi, X).R=0$, then we have

$$\bar{C}(\xi, X)R(Y, Z)U - R(\bar{C}(\xi, X)Y,Z)U - R(Y,\bar{C}(\xi, X)Z)U - \bar{C}(Y, Z)R(\xi, X)U=0. \quad (6.1)$$

By virtue of equation (2.21) above equation reduces to

$$\left[-\varepsilon + \frac{r}{n(n-1)}\right][g(X, R(Y, Z)U)\xi - \varepsilon g(X, R(Y, Z)U)\xi - g(X, Y)R(\xi, Z)U - g(X, Z)R(Y, \xi)U] + \left[\varepsilon + \frac{r\varepsilon}{n(n-1)}\right][\eta(R(Y, Z)U)X + \eta(Y)R(X, Z)U + \eta(Z)R(Y, Z)U] + \varepsilon g(X, U)\bar{C}(Y, Z)\xi - \varepsilon\eta(U)R(Y, Z)X=0. \quad (6.2)$$

Taking the inner product of above equation with ξ and using equation (2.2) and (2.3), we get

$$\left[-\varepsilon + \frac{r}{n(n-1)}\right][g(X, R(Y, Z)U) - \varepsilon g(X, R(Y, Z)U) - g(X, Y)\eta(R(\xi, Z)U) - g(X, Z)\eta(R(Y, \xi)U)] + \left[\varepsilon + \frac{r\varepsilon}{n(n-1)}\right]\begin{bmatrix}\eta(R(Y, Z)U)\eta(X) + \eta(Y)\eta(R(X, Z)U) \\ +\eta(Z)\eta(R(Y, Z)U)\end{bmatrix} + \varepsilon g(X, U)\eta(\bar{C}(Y, Z)\xi) - \varepsilon\eta(U)\eta(R(Y,Z)X)=0. \quad (6.3)$$

Using equations (2.16) and (2.23), X=$\xi$ in above equation, we obtain

$$\left[-\varepsilon + \frac{r}{n(n-1)}\right][-g(X, U)\eta(Y) - g(Y, U)\eta(Z))$$
$$-K_2\{g(Z, U)\eta(Y) - 2\varepsilon\eta(U)\eta(Y)\eta(Z) + g(Y, U)\eta(Z)\}$$
$$-\varepsilon g(X, R(Y,Z)U) - g(X,Y)\eta(R(\xi,Z)U) - g(X,Z)\eta(R(Y,\xi)U)]$$
$$+\left[\varepsilon + \frac{r\varepsilon}{n(n-1)}\right] K_2[(1+\varepsilon)g(Z, U)\eta(Y) - 2\varepsilon g(Y,U)\eta(Z)] = 0. \quad (6.4)$$

In view of equation Y=Z=$\xi$, above equation reduces to

$$\left[\varepsilon + \frac{r\varepsilon}{n(n-1)}\right] K_2\eta(U) = 0. \quad (6.5)$$

In virtue of equation (2.19), in putting U=$\xi$, we get

$$\left[\varepsilon + \frac{r\varepsilon}{n(n-1)}\right]\left[\varepsilon - \frac{r}{n(n-1)}\right] = 0, \text{ where } r = n(\phi X - \lambda X)$$
$$\lambda > 0,$$

which shows that $\lambda$ is either steady or expanding. Thus we can state as follows -

**Theorem (6.1):** Ricci Soliton in ($\varepsilon$)-para Sasakian manifolds with $\xi$ as space–like vector field satisfying $\bar{C}(\xi, X).R=0$, is steady or expanding.

## 7. Ricci Soliton in ($\varepsilon$)-Para Sasakian Satisfying R ($\xi$, X).$\bar{C}$ =0.

Let $R(\xi, X).\bar{C}=0$, then we have

$$R(\xi, X)\bar{C}(Y, Z)U - \bar{C}(R(\xi, X)Y, Z)U - \bar{C}(Y, R(\xi,X)Z)U - \bar{C}(Y, Z)R(\xi,X)U = 0. \quad (7.1)$$

By virtue of equation (2.14) above equation reduces to

$$[-\varepsilon g(X, \bar{C}(Y,Z)U)\xi + \varepsilon\eta(\bar{C}(Y, Z)U)X] + \varepsilon g(X, Y)\bar{C}(\xi, Z)U$$
$$-\varepsilon\eta(Y)\bar{C}(X, Z)U + \varepsilon g(X, Z)\bar{C}(Y, \xi)U) - \varepsilon\eta(Z)\bar{C}(Y, X)U$$
$$+\varepsilon g(X, U)\bar{C}(Y, Z)\xi - \varepsilon\eta(U)\bar{C}(Y, Z)X] = 0. \quad (7.2)$$

Taking the inner product of above equation with $\xi$, and using equation (2.2) and (2.3), we get

$$[\varepsilon g(X, \bar{C}(Y, Z)U) - \varepsilon\eta(X)\eta(\bar{C}(Y, Z)U) + g(X, Y)\eta(\bar{C}(\xi, Z)U) - \eta(Y)\eta(\bar{C}(X, Z)U)$$
$$+ g(X, Z)\,\eta(\bar{C}(Y, \xi)U) - \eta(Z)\eta(\bar{C}(Y, X)U)$$
$$+g(X, U)\eta(\bar{C}(Y, Z)\xi) - \eta(U)\eta(\bar{C}(Y, Z)X)] = 0. \quad (7.3)$$

Using equation (2.24) in above equation, we obtain

$$\varepsilon g(X, \bar{C}(Y, Z)U) = \left[\frac{r}{n(n-1)} + \varepsilon\right][(1-\varepsilon)\{g(X, Y)\eta(Z) + g(X, Z)\eta(Y)\}\eta(U)$$
$$-2g(Y, U)\eta(X)\eta(Z) + g(X, Y)g(Z, U) - g(X, Z)g(Y, U)]. \quad (7.4)$$

In view of equation (2.21), in X=$\xi$ above equation reduces to

$$\eta(\bar{C}(Y, Z)U) = \left[\frac{r}{n(n-1)} + \varepsilon\right][(1-\varepsilon)\{\eta(U)\eta(Y)\eta(Z) + (\varepsilon-1)\varepsilon\eta(Z)\eta(Y)\}\eta(U)$$
$$-2g(Y, U)\eta(Z) + \varepsilon g(Z, U)\eta(Y) - g(Y, U)\eta(Z)]. \quad (7.5)$$

Now, putting Y = U = $\xi$, in above equation and by use equations (2.2), (2.3), (2.4) and (2.14), we obtain

$$\left[\frac{r}{n(n-1)} + \varepsilon\right](1+3\varepsilon)\eta(Z)=0. \quad (7.6)$$

In view of equation (2.19), Z=$\xi$, we get

$$\left[\frac{\lambda}{(n-1)} + \varepsilon\right](1+3\varepsilon)=0,$$
$$\lambda = -\varepsilon(n-1)$$

(7.7)

Now, suppose $\xi$ is space –like vector field in ($\varepsilon$)-para Sasakian manifolds, then from equation (7.7), we obtain

$$\lambda < 0,$$

which shows that $\lambda$ is shrinking. Thus we can state as follows -

**Theorem (7.1):** Ricci Soliton in ($\varepsilon$)-para Sasakian manifolds with $\xi$ as space–like vector field satisfying $R(\xi, X).\bar{C} = 0$, is shrinking.

Again if we assume vector field $\xi$ as time–like vector field in ($\varepsilon$)-para Sasakian manifolds then, in view of equation (7.7), we obtain

$$\lambda > 0,$$

which shows that $\lambda$ is expanding. Thus we can state as follows -

**Theorem (7.2)** Ricci Solitons in ($\varepsilon$)-para Sasakian manifolds admitting $\xi$ as time-like vector field satisfying $R(\xi, X).\bar{C}=0$, is expanding.

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