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Impulsive fractional Hahn difference equations with anti-periodic boundary conditions

Yawen Yan[1], Chengmin Hou*
[1]*Department of Mathematics, Yanbian University, Yanji, 133002, P.R. China*
**The corresponding author*

Abstract: This article studies the existence of solutions for impulsive fractional Hahn difference equation with anti-periodic boundary conditions. The existence of solutions are proved by using Leray-Schauder nonlinear alternative, Boyd and Wong fixed point theorem and Rothe fixed point theorem. Illustrative examples are also presented to show the applicability of our result.
Keywords: quantum calculus; impulsive fractional Hahn difference equations; existence;

1. Introduction

In recent years, Hahn fractional calculus had a remarkable development as shown by many authors (see [1-9]). In [1], Hahn combined the classical Jackson q-difference operator and forward difference operator to construct a Hahn difference operator $D_{q,\omega}$, where $q \in (0,1)$ and $\omega > 0$ are fixed. The Hahn difference operator is defined as follows:

$$D_{q,\omega} f(t) = \frac{f(qt+\omega) - f(t)}{t(q-1)+\omega}, \; t \neq \frac{\omega}{1-q}.$$

It was applied to construct the families of orthogonal polynomial and investigate some approximation problems (see [1] and the references therein). We find that

$$D_{q,\omega} f(t) = \begin{cases} \Delta_\omega f(t), \; if \; q = 1, \\ D_q f(t), \; if \; \omega = 0, \\ f'(t), \quad if \; q = 1, \omega \to 0. \end{cases}$$

In [7], the new notions of right reverse of the Hahn operator has been established and their basic properties were obtained. Also, the existence and uniqueness results for initial value problems of first and second-order impulsive q_k, ω_k -Hahn difference equations were studied.

Impulsive differential equations with anti-periodic boundary conditions [10] are used to study the developmental processes that are subject to sudden changes in their state. Due to its abundant theory [11] and applicability in various fields of science and technology, this subject has been highly valued by researchers. It provided a natural framework for mathematical modeling of many physical phenomena that occurring in the area of mechanics, ecology, medicine, biology, and electrical engineering.

To the best of the author knowledge, the existence of anti-periodic boundary problems for impulsive fractional Hahn difference equations has not been well studied till now. We will fill this gap in the literature. The main purpose of this paper is to investigate the existence of solutions of an impulsive fractional Hahn difference equation with anti-periodic boundary conditions given by

$$\begin{cases} {}_{t_k}^{c}D_{q_k,\omega}^{\alpha_k} x(t) = f(t, x(t)), \quad t \in J_k \subseteq [0,T], t \neq t_k, \\ \Delta x(t_k) := x(t_k^+) - x(t_k) = \varphi_k({}_{t_{k-1}}I_{q_{k-1},\omega}^{\beta_{k-1}} x(t_k)), \; k = 1,2,\ldots,m, \\ {}_{t_k}D_{q_k,\omega} x(t_k^+) - {}_{t_{k-1}}D_{q_{k-1},\omega} x(t_k) = \varphi_k^*({}_{t_{k-1}}I_{q_{k-1},\omega}^{\gamma_{k-1}} x(t_k)), \; k = 1,2,\ldots,m, \\ x(0) = x(T), \; {}_0D_{q_0,\omega} x(0) = -{}_m D_{q_m,\omega} x(T), \end{cases} \tag{1}$$

where $0=t_0<t_1<\cdots<t_m<t_{m+1}=T$, ${}_{t_k}^{c}D_{q_k,\omega}^{\alpha_k}$ denotes the Caputo q_k,ω-fractional derivative of order α_k on J_k, $1<\alpha_k\le 2$, $0<q_k<1$, $\omega>0$, $J_k=(t_k,t_{k+1}]$, $J_0=(0,t_1]$, $k=1,2,\ldots,m$, $J=[0,T]$, $J'=J\setminus\{t_1,t_2,\ldots,t_m\}$, $f\in C(J\times R,R)$, $\varphi_k,\varphi_k^*\in C(R,R)$, $k=1,2,\ldots,m$, ${}_{t_k}I_{q_k,\omega}^{\beta_k}$, ${}_{t_k}I_{q_k,\omega}^{\gamma_k}$ denotes the Riemann-Liouville q_k,ω-fractional integral of orders $\beta_k,\gamma_k>0$ on J_k, $k=1,2,\ldots,m-1$.

The paper is organized as follows. Section 2 contains the basic definitions and properties of fractional q,ω-calculus, which will be used in the later section. Then, in Section 3 the main results is enunciated. Some illustrative examples for the existence and uniqueness results presented in Section 4.

2. Preliminaries

First of all, we recall some basic concepts of q,ω-calculus [8,9].

Let $q\in(0,1)$, $\omega>0$, $\omega_0=\omega/(1-q)$, and the *q*-shifting operator as $(n-m)_a^{(0)}=1, (n-m)_a^{(k)}=\prod_{i=0}^{k-1}(n-{}_a\Phi_q^i(m)), k\in N\cup\{\infty\}, (n-m)_a^{(\gamma)}=\prod_{i=0}^{\infty}\frac{n-{}_a\Phi_q^i(m)}{n-{}_a\Phi_q^{i+\gamma}(m)}, \gamma\in R,$

where ${}_a\Phi_q^{\gamma}(m)=q^{\gamma}m+(1-q^{\gamma})a$.

For $\alpha\in R\setminus\{0,-1,-2,\ldots\}$, the *q*-gamma function is as follows:

$$\Gamma_q(\alpha+1)=[\alpha]_q\Gamma_q(\alpha)=[\alpha]_q(1-q)^{(\alpha-1)}/(1-q)^{\alpha-1}.$$

Let $f:I\to R$, Hahn difference operator is defined by

$$D_{q,\omega}f(t)=\begin{cases}\dfrac{f(qt+\omega)-f(t)}{qt+\omega-t}, & if\ t\neq\omega_0,\\ f'(t), & if\ t=\omega_0,\end{cases}$$

and the fractional q,ω-derivative of Riemann-Liouville type by

$${}_aD_{q,\omega}^{\alpha}f(t)=\begin{cases}({}_aI_{q,\omega}^{-\alpha}f)(t), & \alpha<0,\\ f(t), & \alpha=0,\\ (D_{q,\omega}^{\lceil\alpha\rceil}\,{}_aI_{q,\omega}^{\lceil\alpha\rceil-\alpha}f)(x), & \alpha>0.\end{cases}$$

The fractional q,ω-derivative of Caputo type is

$${}_a^cD_{q,\omega}^{\alpha}f(t)=\begin{cases}({}_aI_{q,\omega}^{-\alpha}f)(t), & \alpha<0,\\ f(t), & \alpha=0,\\ (D_{q,\omega}^{\lceil\alpha\rceil}\,{}_aI_{q,\omega}^{\lceil\alpha\rceil-\alpha}f)(x), & \alpha>0,\end{cases}$$

where $\lceil\alpha\rceil$ denotes the smallest integer greater or equal to α.

Definition 2.1 *Let $v\ge 0$ and f be a function defined on $[a,b]$. Hahn's fractional integration of Riemann-Liouville type is given by $({}_aI_{q,\omega}^{0}f)(t)=f(t)$ and*

$${}_aI_{q,\omega}^{v}f(t)=\frac{1}{\Gamma_q(v)}\int_a^t(t-{}_a\Phi_q(s))_{\omega_0}^{(v-1)}f(s)d_{q,\omega}s,\quad v>0,\ t\in[a,b].$$

From [9], we have the following formulas:

$${}_aD_{q,\omega}^{\alpha}((x-a)_{\omega_0}^{(\lambda)})=\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\alpha+1)}(x-a)_{\omega_0}^{(\lambda-\alpha)},\quad(\omega_0<a<x<b),$$

$$ {}_aI^{\alpha}_{q,\omega}((x-a)^{(\lambda)}_{\omega_0}) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)}(x-a)^{(\lambda+\alpha)}_{\omega_0}, \quad (\omega_0 < a < x < b), $$

$$ ({}_aI^{\alpha}_{q,\omega}1)(x) = \frac{1}{\Gamma_q(\alpha+1)}(x-a)^{(\alpha)}_{\omega_0}, \quad (\omega_0 < a < x < b). $$

Definition 2.2 ([12],[13])*Let E be a Banach space and let $A: E \to E$ be a mapping. A is said to be a nonlinear contraction if there exists a continuous non-decreasing function $\psi: R^+ \to R^+$ such that $\psi(0)=0$ and $\psi(\alpha) < \alpha$ for all $\alpha > 0$ with the property:*

$$ \|Ax - Ay\| \le \psi(\|x-y\|), \forall x, y \in E. $$

Lemma 2.1*Let $\alpha, \beta \in R^+$, Hahn's fractional integration has the following semi-group property: $({}_aI^{\beta}_{q,\omega}\,{}_aI^{\alpha}_{q,\omega} f)(t) = ({}_aI^{\alpha+\beta}_{q,\omega} f)(t), \quad (\omega_0 < a < x < b).$*

Lemma 2.2*Let $f(t)$ be a function defined on an interval (ω_0, b) and $\alpha \in R^+$. Then the following is valid:*

$$ (D^{\alpha}_{q,\omega}\,{}_aI^{\alpha}_{q,\omega} f)(t) = f(t), \quad (\omega_0 < a < x < b). $$

Lemma 2.3*Let $\alpha \in (N-1, N]$. Then, for some constants $C_i \in R, i = 1,2,\ldots N$. the following equality holds:*

$$ ({}_aI^{\alpha}_{q,\omega}\,{}_aD^{\alpha}_{q,\omega} f)(t) = f(t) + C_1(t-a)^{(\alpha-1)}_{\omega_0} + \ldots + C_N(t-a)^{(\alpha-N)}_{\omega_0}. $$

Lemma 2.4 *If $f(t)$ is defined and finite, then for $v > 0$ with $N-1 < v \le N$,*

$$ D^{v}_{q,\omega} f(t) = \frac{1}{\Gamma_q(-v)} \int_a^t (t - {}_{\omega_0}\Phi_q(s))^{(-v-1)}_{\omega_0} f(s) d_{q,\omega}s. $$

Lemma 2.5 *Let $\alpha \in (N-1, N]$. Then, for some constants $C_i \in R, i = 1,2,\ldots N-1$, the following equality holds:*

$$ ({}_aI^{\alpha}_{q,\omega}\,{}^c_aD^{\alpha}_{q,\omega} f)(t) = f(t) - \sum_{k=0}^{N-1} \frac{D^k_{q,\omega} f(a)}{\Gamma_q(k+1)}(t-a)^{(k)}_{\omega_0}. $$

Lemma 2.6*Let $h \in C(J, R)$. Then the unique solution of*

$$ \begin{cases} {}^c_{t_k}D^{\alpha_k}_{q_k,\omega} x(t) = h(t), \quad t \in J_k \subseteq [0,T], t \ne t_k, \\ \Delta x(t_k) := x(t_k^+) - x(t_k) = \varphi_k({}_{t_{k-1}}I^{\beta_{k-1}}_{q_{k-1},\omega} x(t_k)), \ k = 1,2,\ldots,m, \\ {}_{t_k}D_{q_k,\omega} x(t_k^+) - {}_{t_{k-1}}D_{q_{k-1},\omega} x(t_k) = \varphi_k^*({}_{t_{k-1}}I^{\gamma_{k-1}}_{q_{k-1},\omega} x(t_k)), \ k = 1,2,\ldots,m, \\ x(0) = x(T), \ {}_0D_{q_0,\omega} x(0) = -{}_{t_m}D_{q_m,\omega} x(T), \end{cases} \tag{1'} $$

is given by

$$ \begin{aligned} x(t) = & -\frac{1}{2}\sum_{i=1}^{m}\left[{}_{t_{i-1}}I^{\alpha_{i-1}}_{q_{i-1},\omega} h(t_i) + \varphi_i({}_{t_{i-1}}I^{\beta_{i-1}}_{q_{i-1},\omega} x(t_i))\right] \\ & -\frac{1}{2}\sum_{i=1}^{m}(T-t_i)\left\{{}_{t_{i-1}}I^{\alpha_{i-1}-1}_{q_{i-1},\omega} h(t_i) + \varphi_i^*({}_{t_{i-1}}I^{\gamma_{i-1}}_{q_{i-1},\omega} x(t_i))\right\} - \frac{1}{2}{}_{t_m}I^{\alpha_m}_{q_m,\omega} h(T) \\ & + (t - \frac{T}{2})\left[-\frac{1}{2}\sum_{i=1}^{m}\left\{{}_{t_{i-1}}I^{\alpha_{i-1}-1}_{q_{i-1},\omega} h(t_i) + \varphi_i^*({}_{t_{i-1}}I^{\gamma_{i-1}}_{q_{i-1},\omega} x(t_i))\right\} - \frac{1}{2}{}_{t_m}I^{\alpha_m-1}_{q_m,\omega} h(T)\right] \\ & + \sum_{i=1}^{k}\left[{}_{t_{i-1}}I^{\alpha_{i-1}}_{q_{i-1},\omega} h(t_i) + \varphi_i({}_{t_{i-1}}I^{\beta_{i-1}}_{q_{i-1},\omega} x(t_i))\right] \\ & + \sum_{i=1}^{k}(t-t_i)\left[{}_{t_{i-1}}I^{\alpha_{i-1}-1}_{q_{i-1},\omega} h(t_i) + \varphi_i^*({}_{t_{i-1}}I^{\gamma_{i-1}}_{q_{i-1},\omega} x(t_i))\right] + {}_{t_k}I^{\alpha_k}_{q_k,\omega} h(t), \end{aligned} \tag{2} $$

Proof For $t \in J_0$, applying ${}_{t_0}I^{\alpha_0}_{q_0,\omega}$ from t_0 to t in the first equation of (1) and using Lemma 2.5, we have

$$ ({}_{t_0}I^{\alpha_0}_{q_0,\omega}\,{}^c_{t_0}D^{\alpha_0}_{q_0,\omega} x)(t) = x(t) - x(0) - t\,{}_{t_0}D_{q_0,\omega} x(t_0) = {}_{t_0}I^{\alpha_0}_{q_0,\omega} h(t), $$

which leads to

$$x(t)=C_0+C_1t+{}_{t_0}I_{q_0,\omega}^{\alpha_0}h(t), \tag{3}$$

where $C_0=x(0)$ and $C_1={}_0D_{q_0,\omega}x(0)$. For $t=t_1$, we obtain

$$x(t_1)=C_0+C_1t_1+{}_{t_0}I_{q_0,\omega}^{\alpha_0}h(t_1), \text{ and } {}_{t_0}D_{q_0,\omega}x(t_1)={}_{t_0}I_{q_0,\omega}^{\alpha_0-1}h(t_1). \tag{4}$$

For $t\in J_1$, on application of ${}_{t_1}I_{q_1,\omega}^{\alpha_1}$ to (1) and using the above process, we get

$$x(t)=x(t_1^+)+(t-t_1)_{t_1}D_{q_1,\omega}x(t_1^+)={}_{t_1}I_{q_1,\omega}^{\alpha_1}h(t). \tag{5}$$

The impulsive condition implies that

$x(t)=C_0+C_1t+[{}_{t_0}I_{q_0,\omega}^{\alpha_0}h(t_1)+\varphi_1({}_{t_0}I_{q_0,\omega}^{\beta_0}x(t_1))]+(t-t_1)[{}_{t_0}I_{q_0,\omega}^{\alpha_0-1}h(t_1)+\varphi_1^*({}_{t_0}I_{q_0,\omega}^{\gamma_0}x(t_1))]+{}_{t_1}I_{q_1,\omega}^{\alpha_1}h(t).$ In the same ways, for $t\in J_2$, we get

$$\begin{aligned}x(t)=C_0+C_1t&+[{}_{t_0}I_{q_0,\omega}^{\alpha_0}h(t_1)+\varphi_1({}_{t_0}I_{q_0,\omega}^{\beta_0}x(t_1))]+[{}_{t_1}I_{q_1,\omega}^{\alpha_1}h(t_2)+\varphi_2({}_{t_1}I_{q_1,\omega}^{\beta_1}x(t_2))]\\&+(t-t_1)\{{}_{t_0}I_{q_0,\omega}^{\alpha_0-1}h(t_1)+\varphi_1^*({}_{t_0}I_{q_0,\omega}^{\gamma_0}x(t_1))\}\\&+(t-t_2)[{}_{t_1}I_{q_1,\omega}^{\alpha_1-1}h(t_2)+\varphi_2^*({}_{t_1}I_{q_1,\omega}^{\gamma_1}x(t_2))]+{}_{t_2}I_{q_2,\omega}^{\alpha_2}h(t).\end{aligned}$$

Repeating this process, for $t\in J_k\subseteq J$, $k=1,2,\ldots,m-1$, we find that

$$\begin{aligned}x(t)=C_0+C_1t&+\sum_{i=1}^{k}[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}}h(t_i)+\varphi_i({}_{t_{i-1}}I_{q_{i-1},\omega}^{\beta_{i-1}}x(t_i))]\\&+\sum_{i=1}^{k}(t-t_i)\{{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}h(t_i)+\varphi_i^*({}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}x(t_i))\}+{}_{t_k}I_{q_k,\omega}^{\alpha_k}h(t).\end{aligned} \tag{6}$$

From (6), we get

$$\begin{aligned}x(T)=C_0+C_1T&+\sum_{i=1}^{m}[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}}h(t_i)+\varphi_i({}_{t_{i-1}}I_{q_{i-1},\omega}^{\beta_{i-1}}x(t_i))]\\&+\sum_{i=1}^{m}(T-t_i)\{{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}h(t_i)+\varphi_i^*({}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}x(t_i))\}+{}_{t_m}I_{q_m,\omega}^{\alpha_m}h(T)\end{aligned}$$

and

$${}_{t_k}D_{q_k,\omega}x(t)=C_1+\sum_{i=1}^{k}[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}h(t_i)+\varphi_i^*({}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}x(t_i))]+{}_{t_k}I_{q_k,\omega}^{\alpha_k-1}h(t),$$

which implies ${}_{t_0}D_{q_0,\omega}x(0)=C_1$ and

$${}_{t_m}D_{q_m,\omega}x(T)=C_1+\sum_{i=1}^{m}[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}h(t_i)+\varphi_i^*({}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}x(t_i))]+{}_{t_m}I_{q_m,\omega}^{\alpha_m-1}h(T).$$

By the boundary condition of (1) we obtain

$$\begin{aligned}C_0=-\frac{1}{2}C_1T&-\frac{1}{2}\sum_{i=1}^{m}[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}}h(t_i)+\varphi_i({}_{t_{i-1}}I_{q_{i-1},\omega}^{\beta_{i-1}}x(t_i))]\\&-\frac{1}{2}\sum_{i=1}^{m}(T-t_i)\{{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}h(t_i)+\varphi_i^*({}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}x(t_i))\}-\frac{1}{2}{}_{t_m}I_{q_m,\omega}^{\alpha_m}h(T)\end{aligned}$$

and

$$C_1=-\frac{1}{2}\sum_{i=1}^{m}\{{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}h(t_i)+\varphi_i^*({}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}x(t_i))\}-\frac{1}{2}{}_{t_m}I_{q_m,\omega}^{\alpha_m-1}h(T).$$

Substituting the values C_0 and C_1 in (6) yields the solution (2).

3. Main results

In the following, our discussion is based on the classic Banach space

$$PC(J,R)=\{x:J\to R: x(t)\in C(J'),\ x(t_k^+) \text{ and } x(t_k^-) \text{ exist, and } x(t_k^-)=x(t_k),$$
$$k=1,2,\dots m\}$$

with the norm $\|x\|_{PC}=\sup\{|x(t)|:t\in J\}$.

Based on the lemma 2.6, we define the operator $A:PC(J,R)\to PC(J,R)$ by

$$\begin{aligned}Ax(t)=&-\frac{1}{2}\sum_{i=1}^{m}\left[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}}f(t_i,x(t_i))+\varphi_i({}_{t_{i-1}}I_{q_{i-1},\omega}^{\beta_{i-1}}x(t_i))\right]\\&-\frac{1}{2}\sum_{i=1}^{m}(T-t_i)\left\{{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}f(t_i,x(t_i))+\varphi_i^*({}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}x(t_i))\right\}-\frac{1}{2}{}_{t_m}I_{q_m,\omega}^{\alpha_m}f(T,x(T))\\&+(t-\frac{T}{2})\left[-\frac{1}{2}\sum_{i=1}^{m}\left\{{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}f(t_i,x(t_i))+\varphi_i^*({}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}x(t_i))\right\}-\frac{1}{2}{}_{t_m}I_{q_m,\omega}^{\alpha_m-1}f(T,x(T))\right]\\&+\sum_{i=1}^{k}\left[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}}f(t_i,x(t_i))+\varphi_i({}_{t_{i-1}}I_{q_{i-1},\omega}^{\beta_{i-1}}x(t_i))\right]\\&+\sum_{i=1}^{k}(t-t_i)\left[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}f(t_i,x(t_i))+\varphi_i^*({}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}x(t_i))\right]\\&+{}_{t_k}I_{q_k,\omega}^{\alpha_k}f(t,x(t)),\end{aligned}\tag{7}$$

where

$${}_aI_{q,\omega}^{p}f(u,x(u))=\frac{1}{\Gamma_q(p)}\int_a^u {}_a(u-{}_a\Phi_q(s))_{\omega_0}^{(p-1)}f(s,x(s))d_{q,\omega}s,$$

$p\in\{\alpha_0,\dots\alpha_m,\alpha_0-1,\dots\alpha_m-1,\beta_0,\dots\beta_{m-1},\gamma_0,\dots\gamma_{m-1}\}$, $q\in\{q_0,\dots,q_m\}$, $a\in\{t_0,\dots t_m\}$ and $u\in\{t,t_1,\dots t_m,T\}$.

Our first existence result based on Leray-Schauder nonlinear alternative(Specific content can be found in the literature [13]).

For convenience, we set the notations:

$$\Omega_1=\frac{3}{2}\sum_{i=1}^{m+1}\frac{(t_i-t_{i-1})_{\omega_0}^{(\alpha_{i-1})}}{\Gamma_{q_{i-1},\omega}(\alpha_{i-1}+1)}+\frac{3}{2}\sum_{i=1}^{m}\frac{(T-t_i)(t_i-t_{i-1})_{\omega_0}^{(\alpha_{i-1}-1)}}{\Gamma_{q_{i-1},\omega}(\alpha_{i-1})}+\frac{T}{4}\sum_{i=1}^{m+1}\frac{(t_i-t_{i-1})_{\omega_0}^{(\alpha_{i-1}-1)}}{\Gamma_{q_{i-1},\omega}(\alpha_{i-1})},$$

$$\Omega_2=\frac{3}{2}\sum_{i=1}^{m}\frac{(t-t_{i-1})_{\omega_0}^{(\beta_{i-1})}}{\Gamma_{q_{i-1},\omega}(\beta_{i-1}+1)},\quad \Omega_3=\frac{3}{2}\sum_{i=1}^{m}\frac{(T-t_i)(t_i-t_{i-1})_{\omega_0}^{(\gamma_{i-1})}}{\Gamma_{q_{i-1},\omega}(\gamma_{i-1}+1)}+\frac{T}{4}\frac{(t_i-t_{i-1})_{\omega_0}^{(\gamma_{i-1})}}{\Gamma_{q_{i-1},\omega}(\gamma_{i-1}+1)}.\tag{8}$$

Theorem 3.1*Assume that*

(H_1) *there exist a continuous non decreasing function* $\kappa:[0,\infty]\to(0,\infty)$ *, a continuous function* $p:J\to R^+$ *with* $p^*=\sup_{t\in J}|p(t)|$ *and constants* $M_1,M_2>0$ *such that* $|f(t,x)|\le p(t)\kappa(|x|),\ \forall(t,x)\in J\times R$,
$|\varphi_k(x)|\le M_1|x|,\ |\varphi_k^*(x)|\le M_2|x|,\ \forall x\in R,$
$k=1,2,\dots,m;$

(H_2) *there exist a constant* $l>0$ *such that*

$$\frac{(1-M_1\Omega_2-M_2\Omega_3)l}{p^*\psi(N)\Omega_1}>1,\quad M_1\Omega_2+M_2\Omega_3<1,\tag{9}$$

where $\Omega_1,\Omega_2,\Omega_3$ *are defined by* (8). *Then the problem* (1) *has at least one solution.*

Proof The proof is carried out in the following steps:

Step 1: A maps bounded sets into bounded sets

For any $\rho > 0$, there exist $K > 0$ such that, for each $x \in B_\rho = \{x \in PC(J,R)\ \|x\|_{PC} \le \rho\}$, we have $\|Ax\|_{PC} \le K$.

For each $t \in J$, we obtain

Step 2:

$$
\begin{aligned}
|Ax(t)| \le & \frac{1}{2}\sum_{i=1}^{m}\left[p^*\kappa(\rho)_{t_{i-1}} I_{q_{i-1},\omega}^{\alpha_{i-1}} 1(t_i) + \rho M_{1\,t_{i-1}} I_{q_{i-1},\omega}^{\beta_{i-1}} 1(t_i)\right] \\
& + \frac{1}{2}\sum_{i=1}^{m}(T-t_i)\left\{p^*\kappa(\rho)_{t_{i-1}} I_{q_{i-1},\omega}^{\alpha_{i-1}-1} 1(t_i) + \rho M_{2\,t_{i-1}} I_{q_{i-1},\omega}^{\gamma_{i-1}} 1(t_i)\right\} + \frac{1}{2} p^*\kappa(\rho)_{t_m} I_{q_m,\omega}^{\alpha_m} 1(T) \\
& + \frac{T}{2}\left[\frac{1}{2}\sum_{i=1}^{m}\left\{p^*\kappa(\rho)_{t_{i-1}} I_{q_{i-1},\omega}^{\alpha_{i-1}-1} 1(t_i) + \rho M_{2\,t_{i-1}} I_{q_{i-1},\omega}^{\gamma_{i-1}} 1(t_i)\right\} + \frac{1}{2} p^*\kappa(\rho)_{t_m} I_{q_m,\omega}^{\alpha_m-1} 1(T)\right] \\
& + \sum_{i=1}^{m}\left[p^*\kappa(\rho)_{t_{i-1}} I_{q_{i-1},\omega}^{\alpha_{i-1}} 1(t_i) + \rho M_{1\,t_{i-1}} I_{q_{i-1},\omega}^{\beta_{i-1}} 1(t_i)\right] \\
& + \sum_{i=1}^{m}(T-t_i)\left[p^*\kappa(\rho)_{t_{i-1}} I_{q_{i-1},\omega}^{\alpha_{i-1}-1} 1(t_i) + \rho M_{2\,t_{i-1}} I_{q_{i-1},\omega}^{\gamma_{i-1}} 1(t_i))\right] \\
& + p^*\kappa(\rho)_{t_m} I_{q_m,\omega}^{\alpha_m} 1(T) \\
= & \ p^*\kappa(\rho)\Omega_1 + \rho M_1\Omega_2 + \rho M_2\Omega_3 := K.
\end{aligned}
$$

A maps bounded sets into equicontinuous sets of $PC(J,R)$.

we suppose $\tau_1, \tau_2 \in J_k\ (k \in \{0,1,2,\ldots,m\})$ with $\tau_1 < \tau_2$, and then

$$
\begin{aligned}
& |Ax(\tau_2) - Ax(\tau_1)| \\
& \le |\tau_2 - \tau_1|\left[\frac{p^*\kappa(\rho)}{2}\sum_{i=1}^{m+1}\frac{(t_i - t_{i-1})_{\omega_0}^{(\alpha_{i-1}-1)}}{\Gamma_{q_{i-1}}(\alpha_{i-1})} + \frac{\rho M_2}{2}\sum_{i=1}^{m}\frac{(t_i - t_{i-1})_{\omega_0}^{(\gamma_{i-1})}}{\Gamma_{q_{i-1}}(\gamma_{i-1}+1)}\right] \\
& \quad + |\tau_2 - \tau_1|\sum_{i=1}^{k}\left|\frac{p^*\kappa(\rho)(t_i - t_{i-1})_{\omega_0}^{(\alpha_{i-1}-1)}}{\Gamma_{q_{i-1}}(\alpha_{i-1})} + \rho M_2\frac{(t_i - t_{i-1})_{\omega_0}^{(\gamma_{i-1})}}{\Gamma_{q_{i-1}}(\gamma_{i-1}+1)}\right| \\
& \quad + \frac{p^*\kappa(\rho)}{\Gamma_{q_k}(\alpha_k)}\left|\int_{t_k}^{\tau_2} {}_{t_k}(\tau_2 -_{\omega_0} \Phi_{q_k}(s))_{\omega_0}^{(\alpha_k-1)}\, {}_{t_k}d_{q_k}s - \int_{t_k}^{\tau_1} {}_{t_k}(\tau_1 -_{\omega_0} \Phi_{q_k}(s))_{\omega_0}^{(\alpha_k-1)}\, {}_{t_k}d_{q_k}s\right|,
\end{aligned}
$$

As $\tau_1 \to \tau_2$, we have $|Ax(\tau_2) - Ax(\tau_1)| \to 0$. By the Arzelá-Ascoli theorem, we can deduce that A is completely continuous.

Step 3: (A priori bounds)

For $\lambda \in (0,1)$, the equation $x = \lambda Ax$ has a solution x. Then, as in the first step and (H_1), we have

$$\|x\|_{PC} \le p^*\psi(\|x\|_{PC})\Omega_1 + \|x\|_{PC} M_1\Omega_2 + \|x\|_{PC} M_2\Omega_3,$$

Thus

$$\frac{(1 - M_1\Omega_2 - M_2\Omega_3)\|x\|_{PC}}{p^*\psi(\|x\|_{PC})\Omega_1} \le 1.$$

By (H_3), there exists l such that $\|x\|_{PC} \ne l$. Defined $U = \{x \in PC(J,R) : \|x\|_{PC} < l\}$. The operator $A : \overline{U} \to PC$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda Ax$ for some $\lambda \in (0,1)$. As a Consequence of nonlinear alternative of Leray-Schauder type, we deduce that A has a fixed point $x \in \overline{U}$ which is a solution of (1). □

In the next existence result, we use Boyd and Wong fixed point theorem.

Lemma 3.1 (Boyd and Wong [12],[13]) *Let E be a Banach space and let $A : E \to E$ be a nonlinear contraction. Then, A has a unique fixed point in E.*

Theorem 3.1 *Assume $f : J \times R \to R$ and $g : [0,1] \to R^+$ are continuous function satisfying the hypothesis:*

(H_3) $|f(t,x) - f(t,y)| \le g(t)\chi^{-1}\ln(1 + |x - y|)$, $|\varphi_k(x) - \varphi_k(y)| \le M_3\chi^{-1}|x - y|$,

$\left|\varphi_k^*(x)-\varphi_k^*(y)\right| \le M_4\chi^{-1}\left|x-y\right|$, *for all* $t\in J$, *and* $x,y\in R$, $M_3, M_4>0$, *the positive constant* χ *is defined by*

$$\begin{aligned}\chi=&\frac{1}{2}\sum_{i=1}^{m}\left[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}}g(t)+M_3\,{}_{t_{i-1}}I_{q_{i-1},\omega}^{\beta_{i-1}}1\right]\\&+\frac{1}{2}\sum_{i=1}^{m}(T-t_i)\left\{{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}g(t)+M_4\,{}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}1\right\}+\frac{1}{2}{}_{t_m}I_{q_m,\omega}^{\alpha_m}g(t)\\&+\frac{T}{2}\left[\frac{1}{2}\sum_{i=1}^{m}\left\{{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}g(t)+M_4\,{}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}1\right\}+\frac{1}{2}{}_{t_m}I_{q_m,\omega}^{\alpha_m-1}g(t)\right]\\&+\sum_{i=1}^{k}\left[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}}g(t)+M_3\,{}_{t_{i-1}}I_{q_{i-1},\omega}^{\beta_{i-1}}1\right]\\&+\sum_{i=1}^{k}(t-t_i)\left[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}g(t)+M_4\,{}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}1\right]+{}_{t_k}I_{q_k,\omega}^{\alpha_k}g(t).\end{aligned}$$

Then the fractional boundary value problem (1) has a unique solution on J.

Proof Define $\psi(\alpha)=\ln(\alpha+1)$ $(\forall\alpha\ge 0)$ is a continuous non-decreasing function. Clearly, it satisfies $\psi(0)=0$ and $\psi(\alpha)<\alpha$, for all $\alpha>0$.

For any $x,y\in PC(J,R)$ and for each $t\in J$, we have

$$\begin{aligned}&\left|(Ax)(t)-(Ay)(t)\right|\le\\&\frac{1}{2}\sum_{i=1}^{m}\left[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}}\left|f(t_i,x)-f(t_i,y)\right|+M_3\chi^{-1}\,{}_{t_{i-1}}I_{q_{i-1},\omega}^{\beta_{i-1}}\ln(\left|x-y\right|+1)(t_i)\right]\\&+\frac{1}{2}\sum_{i=1}^{m}(T-t_i)\left\{{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}\left|f(t_i,x)-f(t_i,y)\right|+M_4\chi^{-1}\,{}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}\ln(\left|x-y\right|+1)(t_i)\right\}\\&+\frac{1}{2}{}_{t_m}I_{q_m,\omega}^{\alpha_m}\left|f(T,x)-f(T,y)\right|\\&+\frac{T}{2}\left[\frac{1}{2}\sum_{i=1}^{m}\left\{{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}\left|f(t_i,x)-f(t_i,y)\right|+M_4\chi^{-1}\,{}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}\ln(\left|x-y\right|+1)(t_i)\right\}\right.\\&\left.+\frac{1}{2}{}_{t_m}I_{q_m,\omega}^{\alpha_m-1}\left|f(T,x)-f(T,y)\right|\right]\\&+\sum_{i=1}^{k}\left[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}}\left|f(t_i,x)-f(t_i,y)\right|+M_3\chi^{-1}\,{}_{t_{i-1}}I_{q_{i-1},\omega}^{\beta_{i-1}}\ln(\left|x-y\right|+1)(t_i)\right]\\&+\sum_{i=1}^{k}(t-t_i)\left[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}\left|f(t_i,x)-f(t_i,y)\right|+M_4\,{}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}\ln(\left|x-y\right|+1)(t_i)\right]\\&+{}_{t_k}I_{q_k,\omega}^{\alpha_k}\left|f(t,x)-f(t,y)\right|\\\le\ &\chi^{-1}\psi(\left\|x-y\right\|)\left\{\frac{1}{2}\sum_{i=1}^{m}\left[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}}g(t)+M_3\,{}_{t_{i-1}}I_{q_{i-1},\omega}^{\beta_{i-1}}1\right]\right.\\&+\frac{1}{2}\sum_{i=1}^{m}(T-t_i)\left\{{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}g(t)+M_4\,{}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}1\right\}+\frac{1}{2}{}_{t_m}I_{q_m,\omega}^{\alpha_m}g(t)\\&+\frac{T}{2}\left[\frac{1}{2}\sum_{i=1}^{m}\left\{{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}g(t)+M_4\,{}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}1\right\}+\frac{1}{2}{}_{t_m}I_{q_m,\omega}^{\alpha_m-1}g(t)\right]\\&+\sum_{i=1}^{k}\left[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}}g(t)+M_3\,{}_{t_{i-1}}I_{q_{i-1},\omega}^{\beta_{i-1}}1\right]\\&\left.+\sum_{i=1}^{k}(t-t_i)\left[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1}g(t)+M_4\,{}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}}1\right]+{}_{t_k}I_{q_k,\omega}^{\alpha_k}g(t)\right\}\\\le\ &\chi^{-1}\psi(\left\|x-y\right\|)\chi\\=\ &\psi(\left\|x-y\right\|).\end{aligned}$$

Then, $\left|(Ax)(t)-(Ay)(t)\right|_{PC} \leq \psi(\|x-y\|)$. So, A is a nonlinear contraction. It follows from Lemma 3.1 that the problem (1) has a unique solution.

Finally, we prove the final result via following fixed point theorem.

Lemma 3.2 ([14],[15])*Suppose that* $A:\overline{\Omega}\to E$ *is a completely continuous operator. Suppose that one of the following condition is satisfied:*

(i) (Altman) $\|Ax-x\|^2 \geq \|Ax\|^2 - \|x\|^2$ *for all* $x\in\partial\Omega$,

(ii) (Rothe) $\|Ax\| \leq \|x\|$ *for all* $x\in\partial\Omega$,

(iii) (Petryshyn) $\|Ax\| \leq \|Ax-x\|$ *for all* $x\in\partial\Omega$.

Then $\deg(I-A,\Omega,\theta)=1$*, and hence* A *has at least one fixed point in* Ω.

Theorem 3.3*Assume that*

(H_4)*the continuous functions* $f: J\times R\to R$ *and* $\varphi_k: R\to R$, $k=1,2,\ldots m.$ *satisfy*

$$\lim_{x\to 0}\frac{f(t,x)}{x}=0,\ \lim_{x\to 0}\frac{\varphi_k(x)}{x}=0 \text{ and } \lim_{x\to 0}\frac{\varphi_k^*(x)}{x}=0,\ k=1,2,\ldots m.$$

Then problem (1) has at least one solution on J.

Proof Let $x\in PC(J,R)$. Taking ε sufficiently small, we can choose two positive constants δ_1 and δ_2 such that $|f(t,x)|<\varepsilon|x|$ whenever $\|x\|_{PC}<\delta_1$ and $\varphi_k(x)<\varepsilon|x|$ whenever $\|x\|_{PC}<\delta_2$ for $k=1,2,\ldots m.$ Setting $\delta=\min\{\delta_1,\delta_2\}$, we define the open ball $B_\delta=\{x\in PC:\|x\|_{PC}<\delta\}$. As in Theorem 3.2, it is clearly that the operator $A:PC\to PC$ is completely continuous. For any $x\in\partial B_\delta$, we have

$$\begin{aligned}|Ax(t)| = &\frac{1}{2}\sum_{i=1}^{m}\left[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}} f(t_i,x(t_i)) + \varphi_i({}_{t_{i-1}}I_{q_{i-1},\omega}^{\beta_{i-1}} x(t_i))\right] \\ &\frac{1}{2}\sum_{i=1}^{m}(T-t_i)\left\{{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1} f(t_i,x(t_i)) + \varphi_i^*({}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}} x(t_i))\right\} + \frac{1}{2}{}_{t_m}I_{q_m,\omega}^{\alpha_m} f(T,x(T)) \\ &+(t-\frac{T}{2})\left[\frac{1}{2}\sum_{i=1}^{m}\left\{{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1} f(t_i,x(t_i)) + \varphi_i^*({}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}} x(t_i))\right\} + \frac{1}{2}{}_{t_m}I_{q_m,\omega}^{\alpha_m-1} f(T,x(T))\right] \\ &+\sum_{i=1}^{k}\left[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}} f(t_i,x(t_i)) + \varphi_i({}_{t_{i-1}}I_{q_{i-1},\omega}^{\beta_{i-1}} x(t_i))\right] \\ &+\sum_{i=1}^{k}(t-t_i)\left[{}_{t_{i-1}}I_{q_{i-1},\omega}^{\alpha_{i-1}-1} f(t_i,x(t_i)) + \varphi_i^*({}_{t_{i-1}}I_{q_{i-1},\omega}^{\gamma_{i-1}} x(t_i))\right] \\ &+{}_{t_k}I_{q_k,\omega}^{\alpha_k} f(t,x(t)) \\ \leq &(\Omega_1+\Omega_2+\Omega_3)\varepsilon|x|.\end{aligned}$$

Setting $\varepsilon \leq (\Omega_1+\Omega_2+\Omega_3)^{-1}$, we deduce that $|Ax|\leq|x|$. So, we have $\|Ax\|\leq\|x\|$. It follows from Lemma3.2(ii) that problem (1) has at least one solution on J.

4. Examples

In this section, we present three examples to illustrate our results.

Example 4.1 Consider the following boundary value problem for impulsive fractional q,ω-difference equations:

$$\begin{cases} {}_{t_k}^{c}D^{\frac{k+3}{k+2}}_{\frac{1}{k^2-3k+4},\omega}x(t)=\frac{1}{(2+t)^2}(\log_e(\frac{|x|}{4}+2)^2), \quad t\in[0,2]\setminus\{t_1,t_2,t_3\}, \\ \Delta x(t_k)=\frac{1}{19+k}\sin({}_{t_{k-1}}I^{(2+(-1)^{k-1})/2}_{\frac{1}{k^2-3k+4},\omega}x(t_k)),\ t_k=\frac{k}{2}, k=1,2,3, \\ {}_{t_k}D_{\frac{1}{k^2-3k+4},\omega}x(t_k^+)-{}_{t_{k-1}}D_{\frac{1}{k^2-3k+4},\omega}x(t_k)=\frac{1}{3+k}{}_{t_{k-1}}I^{(4+(-1)^{k-1})/2}_{\frac{1}{k^2-3k+4},\omega}x(t_k), \\ x(0)=-x(2),\ {}_0D_{\frac{1}{4},\omega}x(0)=-{}_{\frac{3}{2}}D_{\frac{1}{4},\omega}x(2). \end{cases}$$

Here $\alpha_k=(k+3)/(k+2)$, $q_k=1/(k^2-3k+4)$, $k=0,1,2,3$, $\beta_{k-1}=(2+(-1)^{k-1})/2$, $\gamma_{k-1}=(4+(-1)^{k-1})/2$, $t\in[0,2]\setminus\{t_1,t_2,t_3\}$, $m=3$, $T=2$. With these given values, we find that $\Omega_1=5.849251568$, $\Omega_2=1.433715149$, $\Omega_3=2.428688799$, and $|f(t,x)|=\left|\frac{1}{(2+t)^2}(\log_e(\frac{|x|}{4}+2)^2)\right|\le\frac{1}{(2+t)^2}(\frac{|x|}{4}+2)$, $|\varphi_k(y)|=\frac{1}{19+k}|\sin y|\le\frac{1}{20}|y|$, $|\varphi_k^*(z)|=\frac{1}{3+k}|z|\le\frac{1}{4}|z|$. Let $\psi(x)=\frac{x}{4}+2$, $p^*=\sup_{t\in[0,2]}\left|\frac{1}{(2+t)^2}\right|=\frac{1}{4}$, $M_1=\frac{1}{20}$, and $M_2=\frac{1}{4}$, we find that $M_1\Omega_2+M_2\Omega_3=0.6788579572<1$, Also, there exists a constant l such that $l>65.8163183$ satisfying (9). Thus, by Theorem 3.1, the anti-periodic boundary problem (1) has at least one solution.

Example 4.2 Consider the fractional Hahn difference equations with periodic boundary as follows:

$$\begin{cases} {}_{t_k}^{c}D^{(k^2+5)/(k^2+3)}_{1/(k^2-5k+8),\omega}x(t)=\frac{1}{6}e^{-t^2}\ln(|x|+1), \quad t\in[0,5/3]\setminus\{t_1,t_2,t_3,t_4\}, \\ \Delta x(t_k)=\frac{7k}{16}\tan^{-1}({}_{t_{k-1}}I^{(2k+1)/2}_{1/(k^2-5k+8),\omega}x(t_k)),\ t_k=\frac{k}{3}, k=1,2,3,4, \\ {}_{t_k}D_{1/(k^2-5k+8),\omega}x(t_k^+)-{}_{t_{k-1}}D_{1/(k^2-5k+8),\omega}x(t_k)=\frac{1}{4k}\left(\frac{\left|{}_{t_{k-1}}I^{(2k^2-4k+3)/2}_{1/(k^2-5k+8),\omega}x(t_k)\right|}{1+\left|{}_{t_{k-1}}I^{(2k^2-4k+3)/2}_{1/(k^2-5k+8),\omega}x(t_k)\right|}\right), \\ x(0)=-x(5/3),\ {}_0D_{1/8,\omega}x(0)=-{}_{4/3}D_{1/4,\omega}x(5/3). \end{cases}$$

From the equation above, it clear that $\alpha_k=(k^2+5)/(k^2+3)$, $q_k=1/(k^2-5k+8)$, $k=0,1,2,3,4$, $\beta_{k-1}=(2k+1)/2$, $\gamma_{k-1}=(2k^2-4k+3)/2$, $t\in[0,5/3]\setminus\{t_1,t_2,t_3,t_4\}$, $m=4$, $T=5/3$, and $f(t,x)=\frac{1}{6}e^{-t^2}\ln(|x|+1)$, $g(t)=\frac{1}{6}e^{-t^2}$, $\varphi_k(y)=\frac{7k}{16}\tan^{-1}y\le$
$\frac{7}{4}|y|$, $\varphi_k^*(z)=\frac{1}{4k}\frac{|z|}{1+|z|}\le\frac{1}{4}|z|$. Consequently, we can get $\chi=43.744295482$ by computation. Then all the conditions of Theorem 3.2 are satisfied. Hence, by Theorem 3.2, we see that the aforementioned problem has a unique solution.

Example 4.3 Consider the following impulsive anti-periodic problem of a fractional Hahn difference equation:

$$\begin{cases} {}_{t_k}^{c}D_{1/(k^2-5k+8),\omega}^{(k^2+k+3)/(k^2+2)}x(t) = \dfrac{2t}{3t+1}(\sin x(t) - x(t))e^{x^2(t)\cos^4 x(t)}, \quad t\in[0,3/2]\setminus\{t_1,t_2,t_3,t_4,t_5\}, \\ \Delta x(t_k) = \dfrac{kx^4({}_{t_{k-1}}I_{1/(k^2-5k+8),\omega}^{(2k+1)/2}x(t_k)) + 2kx^2(t_k)}{\log(\left|({}_{t_{k-1}}I_{1/(k^2-5k+8),\omega}^{(2k+1)/2}x(t_k))^3(t_k)\right|+2)}),\ t_k=\dfrac{k}{4}, k=1,2,3,4,5, \\ {}_{t_k}D_{1/(k^2-5k+8),\omega}x(t_k^+) - {}_{t_{k-1}}D_{1/(k^2-5k+8),\omega}x(t_k) = \dfrac{k({}_{t_{k-1}}I_{1/(k^2-5k+8),\omega}^{(2k^2-4k+3)/2}x(t_k))^5 + 2k({}_{t_{k-1}}I_{1/(k^2-5k+8),\omega}^{(2k^2-4k+3)/2}x(t_k))^2}{\log(\left|({}_{t_{k-1}}I_{1/(k^2-5k+8),\omega}^{(2k^2-4k+3)/2}x(t_k))^4\right|+2)}, \\ x(0) = -x(3/2),\ {}_0D_{\frac{1}{8},\omega}x(0) = -{}_{\frac{5}{4}}D_{\frac{1}{8},\omega}x(3/2). \end{cases}$$

Let $\alpha_k = (k^2+k+3)/(k^2+2)$, $q_k = 1/(k^2-5k+8)$, $k = 0,1,2,3,4,5$, $\beta_{k-1} = (2k+1)/2$, $\gamma_{k-1} = (2k^2-4k+3)/2$, $t\in[0,3/2]\setminus\{t_1,t_2,t_3,t_4,t_5,\}$, $m=5$, $T=3/2$. With these given values, we find that $\Omega_1 = 4.6842160837249$, $\Omega_2 = 0.351829188$, $\Omega_3 = 1.63136447907$, The functions $f(t,x) = (2t/(3t+1))(\sin x(t) - x(t))e^{x^2(t)\cos^4 x(t)}$, $\varphi_k(x) = (kx^4(t_k)+2kx^2(t_k))/\log(\left|x^3(t_k)\right|+2)$, $\varphi_k^*(z) = (kx^5+2kx^2)/(\log(\left|x^4\right|+2))$. $k=1,2,3,4,5$. Satisfy

$$\lim_{x\to 0}\frac{f(t,x)}{x} = \frac{2t}{3t+1}(\sin x(t) - x(t))e^{x^2(t)\cos^4 x(t)} = 0,$$

$$\lim_{x\to 0}\frac{\varphi_k(x)}{x} = \frac{kx^3(t_k)+2kx(t_k)}{\log(\left|x^3\right|+2)} = 0, \text{ and } \lim_{x\to 0}\frac{\varphi_k^*(x)}{x} = \frac{kx^4+2kx}{\log(\left|x^4\right|+2)} = 0,\ k=1,2,3,4,5.$$

Thus, by Theorem 3.3, problem (1) has at least one solution on $[0,3/2]$.

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