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Existence and Uniqueness of Solutions for Nonlocal Multi-Point Boundary Value Problems of Fractional Order q,ω-Differential Equations

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Abstract: In this paper, we study a class of multi-point boundary value problems for fractional q,ω-difference equations with Caputo fractional q,ω-differential and q,ω-integral. The moving point theorem proves the existence and uniqueness of the solution of the boundary value problem. Firstly, the Banach fixed point theorem is used to prove the existence and uniqueness of the solution; Then, the Leray-Schauder nonlinear theory and Boyd and Wong fixed point theoremis used to study the existence of the solution. Finally, the validity of the result is illustrated.

Keywords: Riemann-Liouville fractional q,ω-integral; Caputo fractional q,ω-derivative; fractional q,ω-differential equation; fixed point theorem; nonlocal multi-point boundary value problem

1. Introduction

The boundary value problem is an important branch in the theory of differential equations. It has a profound physical background and a wide range of theoretical applications [1-3]. In recent years, many scholars have studied the boundary value problems of integer order and fractional q,ω-differential equations[8-10]. However, the multi-point problem of fractional-order q,ω-difference equations with q,ω-integral boundary conditions is rare.

In this paper we are dedicated to considering Hahn fractional differential equations that contain both the integral boundary condition and the multi-point boundary condition:

$$\begin{cases} {}^{c}D_{q,\omega}^{w}x(t)=f(t,x(t)), \\ {}^{c}D_{q,\omega}^{\sigma}x(\xi)=\sum_{i=1}^{n}\alpha_i\,{}^{c}D_{q,\omega}^{\nu}x(\eta_i), \\ x(1)=\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}x(s)d_{q,\omega}s+\sum_{i=1}^{m-2}\gamma_i x(\eta_i), \end{cases} \tag{1.1}$$

where ${}^{c}D_{q,\omega}^{\mu}$ represents the standard Caputo fractional q,ω-derivative of order $\mu\in\{w,\sigma,\nu\}$ satisfying $1<w\le 2$, $0<\sigma,\nu\le 1$. In addition, $\alpha_i,\beta_i,\gamma_i\in R$ and $0<\xi<\eta_1<\eta_2<\ldots<\eta_n<1$, $f:[0,1]\times R\to R$ is a given continuous function.

2. Preliminaries

First of all, we recall some basic concepts of q,ω-calculus [8,9].

Let $q\in(0,1)$, $\omega>0$, $\omega_0=\omega/(1-q)$, and the q-shifting operator as

$$(n-m)_a^{(0)}=1,\,(n-m)_a^{(k)}=\prod_{i=0}^{k-1}(n-{}_a\Phi_q^i(m)),\,k\in N\cup\{\infty\},\,(n-m)_a^{(\gamma)}=\prod_{i=0}^{\infty}\frac{n-{}_a\Phi_q^i(m)}{n-{}_a\Phi_q^{i+\gamma}(m)},\,\gamma\in R,$$

where ${}_a\Phi_q^{\gamma}(m)=q^{\gamma}m+(1-q^{\gamma})a$.

For $\alpha\in R\setminus\{0,-1,-2,\ldots\}$, the q-gamma function is as follows:

$$\Gamma_q(\alpha+1)=[\alpha]_q\Gamma_q(\alpha)=[\alpha]_q(1-q)^{(\alpha-1)}/(1-q)^{\alpha-1}.$$

Let $f:I\to R$, Hahn difference operator is defined by

$$D_{q,\omega} f(t) = \begin{cases} \dfrac{f(qt+\omega) - f(t)}{qt+\omega-t}, & if\ t \neq \omega_0, \\ f'(t), & if\ t = \omega_0, \end{cases}$$

and the fractional q,ω-derivative of Riemann-Liouville type by

$$_a D^{\alpha}_{q,\omega} f(t) = \begin{cases} ({}_a I^{-\alpha}_{q,\omega} f)(t), & \alpha < 0, \\ f(t), & \alpha = 0, \\ (D^{\lceil \alpha \rceil}_{q,\omega}\, {}_a I^{\lceil \alpha \rceil - \alpha}_{q,\omega} f)(x), & \alpha > 0. \end{cases}$$

The fractional q,ω-derivative of Caputo type is

$${}^c_a D^{\alpha}_{q,\omega} f(t) = \begin{cases} ({}_a I^{-\alpha}_{q,\omega} f)(t), & \alpha < 0, \\ f(t), & \alpha = 0, \\ (D^{\lceil \alpha \rceil}_{q,\omega}\, {}_a I^{\lceil \alpha \rceil - \alpha}_{q,\omega} f)(x), & \alpha > 0, \end{cases}$$

where $\lceil \alpha \rceil$ denotes the smallest integer greater or equal to α.

Definition 2.1*Let* $v \geq 0$ *and* f *be a function defined on* $[a,b]$. *Hahn's fractional integration of Riemann-Liouville type is given by* $({}_a I^0_{q,\omega} f)(t) = f(t)$ *and*

$${}_a I^{v}_{q,\omega} f(t) = \frac{1}{\Gamma_q(v)} \int_a^t (t - {}_a\Phi_q(s))^{(v-1)}_{\omega_0} f(s) d_{q,\omega} s, \quad v > 0,\ t \in [a,b].$$

From [7], we have the following formulas:

$${}_a D^{\alpha}_{q,\omega}((x-a)^{(\lambda)}_{\omega_0}) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\alpha+1)} (x-a)^{(\lambda-\alpha)}_{\omega_0}, \quad (\omega_0 < a < x < b),$$

$${}_a I^{\alpha}_{q,\omega}((x-a)^{(\lambda)}_{\omega_0}) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)} (x-a)^{(\lambda+\alpha)}_{\omega_0}, \quad (\omega_0 < a < x < b),$$

$$({}_a I^{\alpha}_{q,\omega} 1)(x) = \frac{1}{\Gamma_q(\alpha+1)} (x-a)^{(\alpha)}_{\omega_0}, \quad (\omega_0 < a < x < b).$$

Lemma 2.1 *Let* $\alpha \in (N-1, N]$. *Then, for some constants* $C_i \in R, i = 1,2,\ldots N-1$, *the following equality holds:*

$$({}_a I^{\alpha}_{q,\omega}\, {}^c_a D^{\alpha}_{q,\omega} f)(t) = f(t) - \sum_{k=0}^{N-1} \frac{D^k_{q,\omega} f(a)}{\Gamma_q(k+1)} (t-a)^{(k)}_{\omega_0}.$$

Lemma 2.2 *Assume that* $h \in C[0,1]$, *then the solution to the fractional differential equation*

$$\begin{cases} {}^c D^{w}_{q,\omega} x(t) = h(t), \\ {}^c D^{\sigma}_{q,\omega} x(\xi) = \sum_{i=1}^{n} \alpha_i\, {}^c D^{v}_{q,\omega} x(\eta_i), \\ x(1) = \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} x(s) d_{q,\omega} s + \sum_{i=1}^{m-2} \gamma_i x(\eta_i), \end{cases} \tag{2.1}$$

is given by

$$x(t)=\int_0^t \frac{(t-{}_0\Phi_q(s))_{\omega_0}^{(w-1)}}{\Gamma_q(w)}h(s)d_{q,\omega}s+\frac{1}{\Delta_2}\left(-\int_0^1 \frac{(1-{}_0\Phi_q(s))_{\omega_0}^{(w-1)}}{\Gamma_q(w)}h(s)d_{q,\omega}s+\sum_{i=1}^{n}\beta_i\int_0^{\eta_i}\frac{(\eta_i-{}_0\Phi_q(s))_{\omega_0}^{(w)}}{\Gamma_q(w+1)}h(s)d_{q,\omega}s\right.$$
$$\left.+\sum_{i=1}^{n}\gamma_i\int_0^{\eta_i}\frac{(\eta_i-{}_0\Phi_q(s))_{\omega_0}^{(w-1)}}{\Gamma_q(w)}h(s)d_{q,\omega}s\right)+\frac{\Gamma_q(2-\sigma)\Gamma_q(2-\nu)(\Delta_3+t\Delta_2)}{\Delta_1\Delta_2}$$
$$\times\left(-\int_0^{\xi}\frac{(\xi-{}_0\Phi_q(s))_{\omega_0}^{(w-\sigma-1)}}{\Gamma_q(w-\sigma)}h(s)d_{q,\omega}s+\sum_{i=1}^{n}\alpha_i\int_0^{\eta_i}\frac{(\eta_i-{}_0\Phi_q(s))_{\omega_0}^{(w-\nu-1)}}{\Gamma_q(w-\nu)}h(s)d_{q,\omega}s\right),$$

where,

$$\Delta_1=\xi^{1-\sigma}\Gamma_q(2-\nu)-\sum_{i=1}^{n}\alpha_i\eta_i^{1-\nu}\Gamma_q(2-\sigma)\neq 0,$$

$$\Delta_2=\sum_{i=1}^{n}\beta_i\eta_i+\gamma_i-1\neq 0, \tag{2.2}$$

$$\Delta_3=\sum_{i=1}^{n}\eta_i\left(\frac{\beta_i\eta_i}{1+q}+\gamma_i\right)-1.$$

Proof By Lemmas 2.1, the following equality holds:

$$x(t)=\frac{1}{\Gamma_q(w)}\int_0^t (t-{}_0\Phi_q(s))_{\omega_0}^{(w-1)}h(s)d_{q,\omega}s+x(0)+tD_{q,\omega}x(0), \tag{2.3}$$

$$\int_0^{\eta_i}x(s)d_{q,\omega}s=\int_0^{\eta_i}\left(\int_0^t\frac{(t-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)}h(\tau)d_{q,\omega}\tau\right)d_qs+x(0)\eta_i+D_{q,\omega}x(0)\frac{\eta_i^2}{1+q}$$
$$=\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)}h(s)d_{q,\omega}s+x(0)\eta_i+D_{q,\omega}x(0)\frac{\eta_i^2}{1+q},\quad i=1,2,\ldots n \tag{2.4}$$

By taking the Caputo fractional derivative ${}^cD_{q,w}^{\nu}$, ${}^cD_{q,w}^{\sigma}$ to both sides of (2.3), we get

$${}^cD_{q,\omega}^{\sigma}x(\xi)=\int_0^{\xi}\frac{(\xi-qs)_{\omega_0}^{(w-\sigma-1)}}{\Gamma_q(w-\sigma)}h(s)d_{q,\omega}s+D_{q,\omega}x(0)\frac{\Gamma_q(2)\xi^{1-\sigma}}{\Gamma_q(2-\sigma)},\ \ i=1,2,\ldots n.$$

$${}^cD_{q,\omega}^{\nu}x(\eta_i)=\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w-\nu-1)}}{\Gamma_q(w-\nu)}h(s)d_{q,\omega}s+D_{q,\omega}x(0)\frac{\eta_i^{1-\nu}}{\Gamma_q(2-\nu)},$$

The boundary condition ${}^cD_{q,\omega}^{\sigma}x(\xi)=\sum_{i=1}^{n}\alpha_i\,{}^cD_{q,\omega}^{\nu}x(\eta_i)$ implies that

$$\int_0^{\xi}\frac{(\xi-qs)_{\omega_0}^{(w-\sigma-1)}}{\Gamma_q(w-\sigma)}h(s)d_{q,\omega}s+D_{q,\omega}x(0)\frac{\xi^{1-\sigma}}{\Gamma_q(2-\sigma)}$$
$$=\sum_{i=1}^{n}\alpha_i\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w-\nu-1)}}{\Gamma_q(w-\nu)}h(s)d_{q,\omega}s+\sum_{i=1}^{n}\alpha_iD_{q,\omega}x(0)\frac{\eta_i^{1-\nu}}{\Gamma_q(2-\nu)}. \tag{2.5}$$

and

$$D_{q,\omega}x(0)=\frac{\Gamma_q(2-\sigma)\Gamma_q(2-\nu)}{\Delta_1}\left(-\int_0^{\xi}\frac{(\xi-qs)_{\omega_0}^{(w-\sigma-1)}}{\Gamma_q(w-\sigma)}h(s)d_{q,\omega}s+\sum_{i=1}^{n}\alpha_i\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w-\nu-1)}}{\Gamma_q(w-\nu)}h(s)d_{q,\omega}s\right),$$

The boundary condition $x(1)=\sum_{i=1}^{n}\beta_i\int_0^{\eta_i}x(s)d_{q,\omega}s+\sum_{i=1}^{n}\gamma_i x(\eta_i)$ implies that

$$\begin{aligned}&\int_0^1\frac{(1-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)}h(s)d_{q,\omega}s+x(0)+D_{q,\omega}x(0)\\&\quad=\sum_{i=1}^{n}\beta_i\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w)}}{\Gamma_q(w+1)}h(s)d_{q,\omega}s+x(0)\sum_{i=1}^{n}\beta_i\eta_i+D_{q,\omega}x(0)\sum_{i=1}^{n}\frac{\eta_i^2\beta_i}{1+q}\\&\quad+\sum_{i=1}^{n}\gamma_i\left(\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)}h(s)d_{q,\omega}s+x(0)+D_{q,\omega}x(0)\eta_i\right).\end{aligned}$$

Inserting the value of $D_{q,\omega}x(0)$ in the above equation, we obtain

$$\begin{aligned}x(0)=&\frac{1}{\Delta_2}\left(-\int_0^1\frac{(1-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)}h(s)d_{q,\omega}s+\sum_{i=1}^{n}\beta_i\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w)}}{\Gamma_q(w+1)}h(s)d_{q,\omega}s\right.\\&\left.+\sum_{i=1}^{n}\gamma_i\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)}h(s)d_{q,\omega}s\right)+\frac{\Delta_3\Gamma_q(2-\nu)\Gamma_q(2-\sigma)}{\Delta_1\Delta_2}\\&\left(-\int_0^{\xi}\frac{(\xi-qs)_{\omega_0}^{(w-\sigma-1)}}{\Gamma_q(w-\sigma)}h(s)d_{q,\omega}s+\sum_{i=1}^{n}\alpha_i\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w-\nu-1)}}{\Gamma_q(w-\nu)}h(s)d_{q,\omega}s\right),\end{aligned}$$

Hence, the solution is (2.2). This completes the proof. □

3. Existence and Uniqueness Results

Let $\kappa=C[0,1]$ be the Banach space of all continuous functions from $[0,1]$ into R endowed with the norm: $\|x\|=\sup\{|x(t)|,t\in[0,1]\}$. In view of Lemma 2.5, we define an operator $\mathrm{A}:\kappa\to\kappa$ by

$$\begin{aligned}(\mathrm{A}x)(t)=&\int_0^t\frac{(t-qs)_{\omega_0}^{(w-1)}f(s,x(s))}{\Gamma_q(w)}d_{q,\omega}s+\frac{1}{\Delta_2}\left(-\int_0^1\frac{(1-qs)_{\omega_0}^{(w-1)}f(s,x(s))}{\Gamma_q(w)}d_{q,\omega}s\right.\\&\left.+\sum_{i=1}^{n}\beta_i\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w)}}{\Gamma_q(w+1)}f(s,x(s))d_{q,\omega}s+\sum_{i=1}^{n}\gamma_i\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)}f(s,x(s))d_{q,\omega}s\right)\\&+\frac{\Gamma_q(2-\sigma)\Gamma_q(2-\nu)(\Delta_3+\Delta_2t)}{\Delta_1\Delta_2}\left(-\int_0^{\xi}\frac{(\xi-qs)_{\omega_0}^{(w-\sigma-1)}}{\Gamma_q(w-\sigma)}f(s,x(s))d_{q,\omega}s\right.\\&\left.+\sum_{i=1}^{n}\alpha_i\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w-\nu-1)}}{\Gamma_q(w-\nu)}f(s,x(s))d_{q,\omega}s\right).\end{aligned}\tag{3.1}$$

Clearly, x is a solution of problem (1.1) if and only if x is a fixed point of the operator A.

For convenience, we set the notations:

$$\begin{aligned}\Theta=&\frac{1}{\Gamma_q(w+1)}+\frac{1}{|\Delta_2|\Gamma_q(w+2)}\left(1+w+\sum_{i=1}^{n}\eta_i^w\big(\eta_i|\beta_i|+(w+1)|\gamma_i|\big)\right)\\&+\frac{\Gamma_q(2-\sigma)\Gamma_q(2-\nu)\big(|\Delta_3|+|\Delta_2|\big)}{|\Delta_1\Delta_2|}\left(\frac{\xi^{w-\sigma}}{\Gamma_q(w-\sigma+1)}+\sum_{i=1}^{n}|\alpha_i|\frac{\eta_i^{w-\nu}}{\Gamma_q(w-\nu+1)}\right),\end{aligned}\tag{3.2}$$

$$\Omega = \frac{1}{\Gamma_q(w)} + \frac{\Gamma_q(2-\sigma)\Gamma_q(2-v)}{|\Delta_1|}\left(\frac{\xi^{w-\sigma}}{\Gamma_q(w-\sigma+1)} + \sum_{i=1}^{n}|\alpha_i|\frac{\eta_i^{w-v}}{\Gamma_q(w-v+1)}\right). \quad (3.3)$$

First, we prove an existence and uniqueness result via Banach's fixed point theorem.

Theorem 3.1 Let $f:[0,1]\times R \to R$ be a continuous function satisfying the Lipschitz condition:
(H_1) there exists a constant $L>0$ such that $|f(t,x)-f(t,y)| \le L|x-y|$, for each $t\in[0,1]$ and $x,y\in R$. Then the fractional boundary value problem (1.1)-(1.2) has a unique solution on $[0,1]$ if

$$L\Theta < 1$$

where Θ is given by (3.2).

Proof Applying Banach's contraction mapping principle, we shall show that A *has a unique fixed point. Let* $\sup\{|f(t,0)|, t\in[0,1]\} = M < \infty$. Choose a constant $\rho > 0$ satisfying $\rho \ge \Theta M(1-\Theta L)^{-1}$. For

$\forall x \in B_\rho = \{x \in \kappa : \|x\| \le \rho\},\ t\in[0,1]$,

$$\begin{aligned} |f(t,x(t))| &\le |f(t,x(t)) - f(t,0)| + |f(t,0)| \\ &\le L\|x\| + M \\ &\le L\rho + M. \end{aligned} \quad (3.5)$$

From (3.1), (3.5), (3.2), we obtain

$$\begin{aligned} |(\mathrm{A}x)(t)| \le & \int_0^t \frac{(t-qs)_{\omega_0}^{(w-1)}|f(s,x(s))|}{\Gamma_q(w)} d_{q,\omega}s + \frac{1}{|\Delta_2|}\Bigg(\int_0^1 \frac{(1-qs)_{\omega_0}^{(w-1)}|f(s,x(s))|}{\Gamma_q(w)} d_{q,\omega}s \\ & + \sum_{i=1}^{n}|\beta_i|\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w)}}{\Gamma_q(w+1)}|f(s,x(s))|d_{q,\omega}s + \sum_{i=1}^{n}|\gamma_i|\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)}|f(s,x(s))|d_{q,\omega}s\Bigg) \\ & + \frac{\Gamma_q(2-\sigma)\Gamma_q(2-v)(|\Delta_3|+|\Delta_2|)}{|\Delta_1\Delta_2|}\Bigg(\int_0^{\xi}\frac{(\xi-qs)_{\omega_0}^{(w-\sigma-1)}}{\Gamma_q(w-\sigma)}|f(s,x(s))|d_{q,\omega}s, \\ & + \sum_{i=1}^{n}|\alpha_i|\int_0^{\eta_i}\frac{(\eta_i-s)_{\omega_0}^{(w-v-1)}}{\Gamma_q(w-v)}|f(s,x(s))|d_{q,\omega}s\Bigg) \\ \le & \Bigg\{\int_0^t \frac{(t-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)} d_{q,\omega}s + \frac{1}{|\Delta_2|}\Bigg(\int_0^1 \frac{(1-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(\omega)} d_{q,\omega}s + \sum_{i=1}^{n}|\beta_i|\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w)}}{\Gamma_q(w+1)}d_{q,\omega}s \\ & + \sum_{i=1}^{n}|\gamma_i|\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)}d_{q,\omega}s\Bigg) + \frac{\Gamma_q(2-\sigma)\Gamma_q(2-v)(|\Delta_3|+|\Delta_2|)}{|\Delta_1\Delta_2|}\Bigg(\int_0^{\xi}\frac{(\xi-qs)_{\omega_0}^{(w-\sigma-1)}}{\Gamma_q(w-\sigma)}d_{q,\omega}s \\ & + \sum_{i=1}^{n}|\alpha_i|\int_0^{\eta_i}\frac{(\eta_i-s)_{\omega_0}^{(w-v-1)}}{\Gamma_q(w-v)}d_{q,\omega}s\Bigg)\Bigg\} \times (L\|x\|+M) \\ = & \Bigg\{\frac{t^w}{\Gamma_q(w+1)} + \frac{1}{|\Delta_2|}\Bigg(\frac{1}{\Gamma_q(w+1)} + \sum_{i=1}^{n}|\beta_i|\frac{\eta_i^{w+1}}{\Gamma_q(\omega+2)} + \sum_{i=1}^{n}|\gamma_i|\frac{\eta_i^{w}}{\Gamma_q(w+1)}\Bigg) \\ & + \frac{\Gamma_q(2-\sigma)\Gamma_q(2-v)(|\Delta_3|+|\Delta_2|)}{|\Delta_1\Delta_2|}\Bigg(\frac{\xi^{w-\sigma}}{\Gamma_q(w-\sigma+1)} + \sum_{i=1}^{n}|\alpha_i|\frac{\eta_i^{w-v}}{\Gamma_q(w-v+1)}\Bigg)\Bigg\} \\ & \times (L\|x\|+M) \\ \le & \Bigg\{\frac{1}{\Gamma_q(w+1)} + \frac{1}{|\Delta_2|\Gamma_q(w+2)}\Bigg([1+w]_q + \sum_{i=1}^{n}\eta_i^{w}\left(|\beta_i|\eta_i + |\gamma_i|[1+\omega]_q\right) + \sum_{i=1}^{n}|\gamma_i|[1+\omega]_q\Bigg) \end{aligned}$$

which

implies $\|\mathrm{A}x\| \le \rho$, So, we have $\mathrm{A}B_\rho \subset B_\rho$.

For $x(t), y(t) \in \kappa$, $\forall t \in [0,1]$, we get

$$
\begin{aligned}
&|(\mathrm{A}x)(t) - (\mathrm{A}y)(t)| \\
&\le \int_0^t \frac{(t-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)} |f(s,x(s)) - f(s,y(s))| d_{q,\omega}s + \frac{1}{|\Delta_2|}\left(\int_0^1 \frac{(1-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)} |f(s,x(s)) - f(s,y(s))| d_{q,\omega}s \right.\\
&+ \sum_{i=1}^n |\beta_i| \int_0^{\eta_i} \frac{(\eta_i - qs)_{\omega_0}^{(w)}}{\Gamma_q(\omega+1)} |f(s,x(s)) - f(s,y(s))| d_{q,\omega}s \\
&\left. + \sum_{i=1}^n |\gamma_i| \int_0^{\eta_i} \frac{(\eta_i - qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)} |f(s,x(s)) - f(s,y(s))| d_{q,\omega}s \right) \\
&+ \frac{\Gamma_q(2-\sigma)\Gamma_q(2-\nu)(|\Delta_3| + |\Delta_2|)}{|\Delta_1 \Delta_2|} \left(\int_0^{\xi} \frac{(\xi - qs)_{\omega_0}^{(w-\sigma-1)}}{\Gamma_q(w-\sigma)} |f(s,x(s)) - f(s,y(s))| d_{q,\omega}s \right.\\
&\left. + \sum_{i=1}^n |\alpha_i| \int_0^{\eta_i} \frac{(\eta_i - s)_{\omega_0}^{(w-\nu-1)}}{\Gamma_q(w-\nu)} |f(s,x(s)) - f(s,y(s))| d_{q,\omega}s \right) \\
&\le L\|x-y\| \left\{ \frac{1}{\Gamma_q(w+1)} + \frac{1}{|\Delta_2|\Gamma_q(w+2)} \left([1+w]_q + \sum_{i=1}^n \eta_i^{\,w} \left(|\beta_i| \eta_i + |\gamma_i| [1+w]_q \right) \right) \right.\\
&\left. + \frac{\Gamma_q(2-\sigma)\Gamma_q(2-\nu)(|\Delta_3| + |\Delta_2|)}{|\Delta_1 \Delta_2|} \left(\frac{\xi^{w-\sigma}}{\Gamma_q(w-\sigma+1)} + \sum_{i=1}^n |\alpha_i| \frac{\eta_i^{\,w-\nu}}{\Gamma_q(w-\nu+1)} \right) \right\} \\
&= L\Theta \|x-y\|.
\end{aligned}
$$

Consequently, $\|(\mathrm{A}x)(t) - (\mathrm{A}y)(t)\| \le L\Theta\|x-y\|$. As $L\Theta < 1$, it follows that the operator A is a contraction. By the Banach's contraction mapping principle, A has a fixed point in B_ρ which is the unique solution of the problem (1.1) on $[0,1]$. This completes the proof.

Second, we give a second existence and uniqueness result based on nonlinear contractions.

Definition 3.1 Let κ be a Banach space and let $\mathrm{A}: \kappa \to \kappa$ be a mapping. A is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\psi: R^+ \to R^+$ such that $\psi(0) = 0$, $\psi(\alpha) < \alpha$ for all $\alpha > 0$ with the property: $\|Ax - Ay\| \le \psi(\|x-y\|),\ \forall x, y \in \kappa$.

Lemma 3.1 (Boyd and Wong[6]). Let κ be a Banach space and let $\mathrm{A}: \kappa \to \kappa$ be a nonlinear contraction. Then, A has a unique fixed point in κ.

Theorem 3.2 Let $f: [0,1] \times R \to R$ be a continuous function satisfying the assumption:

(H_2) $|f(t,x) - f(t,y)| \le g(t)\phi^{-1} \ln(1 + |x-y|)$, for all $t \in [0,1]$ and $\forall x, y \in \kappa$, where $g: [0,1] \to R^+$ is continuous and the positive constant ϕ is defined by

$$
\begin{aligned}
\phi = (1 + \frac{1}{|\Delta_2|}) \int_0^1 \frac{(1-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)} g(s) d_{q,\omega}s + \frac{1}{|\Delta_2|} \sum_{i=1}^n \int_0^{\eta_i} \frac{(\eta_i - qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w+1)} (\omega|\gamma_i| + |\beta_i|(\eta_i - s)) g(s) d_{q,\omega}s \\
+ \frac{\Gamma_q(2-\sigma)\Gamma_q(2-\nu)}{|\Delta_1 \Delta_2|} \left(\int_0^{\xi} \frac{(\xi - qs)_{\omega_0}^{(w-\sigma-1)}}{\Gamma_q(w-\sigma)} g(s) d_{q,\omega}s + \sum_{i=1}^n |\alpha_i| \int_0^{\eta_i} \frac{(\eta_i - qs)_{\omega_0}^{(w-\nu-1)}}{\Gamma_q(w-\nu)} g(s) d_{q,\omega}s \right)
\end{aligned}
$$

Proof Let the operator $\mathrm{A}: \kappa \to \kappa$ be defined as in (3.1), Consider the continuous non decreasing function $\psi: R^+ \to R^+$ defined by $\psi(\alpha) = \ln(\alpha + 1), \forall \alpha \ge 0$, Clearly, the function ψ satisfies $\psi(0) = 0$, $\psi(\alpha) < \alpha$ ($\forall \alpha > 0$). For any $x, y \in \kappa$, $t \in [0,1]$, we have

$$
\begin{aligned}
&|(\mathrm{A}x)(t)-(\mathrm{A}y)(t)| \\
&\leq \int_0^t \frac{(t-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)}|f(s,x(s))-f(s,y(s))|d_{q,\omega}s+\frac{1}{|\Delta_2|}\left(\int_0^1 \frac{(1-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)}|f(s,x(s))-f(s,y(s))|d_{q,\omega}s\right. \\
&\quad+\sum_{i=1}^n|\beta_i|\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w)}}{\Gamma_q(w+1)}|f(s,x(s))-f(s,y(s))|d_{q,\omega}s \\
&\quad\left.+\sum_{i=1}^n|\gamma_i|\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)}|f(s,x(s))-f(s,y(s))|d_{q,\omega}s\right) \\
&\quad+\frac{\Gamma_q(2-\sigma)\Gamma_q(2-v)(|\Delta_3|+|\Delta_2|)}{|\Delta_1\Delta_2|}\left(\int_0^{\xi}\frac{(\xi-qs)_{\omega_0}^{(w-\sigma-1)}}{\Gamma_q(w-\sigma)}|f(s,x(s))-f(s,y(s))|d_{q,\omega}s\right. \\
&\quad\left.+\sum_{i=1}^n|\alpha_i|\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w-v-1)}}{\Gamma_q(w-v)}|f(s,x(s))-f(s,y(s))|d_{q,\omega}s\right) \\
&\leq \phi^{-1}\int_0^t\frac{(t-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)}g(s)\ln(1+|x(s)-y(s)|)d_{q,\omega}s+\frac{\phi^{-1}}{|\Delta_2|}\int_0^1\frac{(1-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)}g(s)\ln(1+|x(s)-y(s)|)d_{q,\omega}s \\
&\quad+\sum_{i=1}^n|\beta_i|\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w)}}{\Gamma_q(w+1)}g(s)\ln(1+|x(s)-y(s)|)d_{q,\omega}s \\
&\quad\left.+\sum_{i=1}^n|\gamma_i|\int_0^{\eta_i}\frac{(\eta_i-qs)_{\omega_0}^{(w-1)}}{\Gamma_q(w)}g(s)\ln(1+|x(s)-y(s)|)d_{q,\omega}s\right) \\
&\quad+\frac{\Gamma_q(2-\sigma)\Gamma_q(2-v)(|\Delta_3|+2|\Delta_2|)}{2|\Delta_1\Delta_2|}\left(\int_0^{\xi}\frac{(\xi-qs)_{\omega_0}^{(w-\sigma-1)}}{\Gamma_q(w-\sigma)}g(s)\ln(1+|x(s)-y(s)|)d_{q,\omega}s,\right. \\
&\quad\left.+\sum_{i=1}^n|\alpha_i|\int_0^{\eta_i}\frac{(\eta_i-s)_{\omega_0}^{(w-v-1)}}{\Gamma_q(w-v)}g(s)\ln(1+|x(s)-y(s)|)d_{q,\omega}s\right) \\
&=\phi^{-1}\psi(\|x-y\|)\phi \\
&=\psi(\|x-y\|).
\end{aligned}
$$

Hence, A is a nonlinear contraction. Therefore, by Lemma 4, the operator A has a unique solution.

4. Examples

4.1 We consider the following nonlocal fractional boundary value problem

$$
\begin{cases}
{}^cD_{q,\omega}^{\frac{3}{2}}x(t)=\dfrac{te^{-\pi t}}{56+e^{-2\pi}}\sin x+\dfrac{e^{-\cos^2 t}}{\sqrt{64+t}}\tan^{-1}x+\dfrac{1}{3},\ \ t\in[0,1] \\
{}^cD_{q,\omega}^{\frac{1}{3}}x(\frac{3}{5})={}^cD_{q,\omega}^{\frac{1}{4}}x(\frac{4}{5})+\frac{1}{2}{}^cD_{q,\omega}^{\frac{1}{4}}x(\frac{6}{7}), \\
x(1)=\dfrac{1}{3}\displaystyle\int_0^{\frac{4}{5}}x(s)d_{q,\omega}s+\dfrac{2}{3}\int_0^{\frac{6}{7}}x(s)d_{q,\omega}s+3x(\frac{4}{5})+\frac{1}{7}x(\frac{6}{7}),
\end{cases}
\tag{4.1}
$$

where $w=\frac{3}{2}$, $q=\frac{1}{2}$, $v=\frac{1}{4}$, $\sigma=\frac{1}{3}$, $\eta_1=\frac{4}{5}$, $\eta_2=\frac{6}{7}$, $\xi=\frac{3}{5}$, $\alpha_1=1$, $\alpha_2=\frac{1}{2}$, $\beta_1=\frac{1}{3}$, $\beta_2=\frac{2}{3}$, $\gamma_1=3$, $\gamma_2=\frac{1}{7}$, $f(t,x)=\dfrac{te^{-\pi t}}{56+e^{-2\pi}}\sin x+\dfrac{e^{-\cos^2 t}}{\sqrt{64+t}}\tan^{-1}x+\dfrac{1}{3}$, Since

$$\begin{aligned}|f(t,x)-f(t,y)| &\le \frac{te^{-\pi t}}{56+e^{-2\pi}}|\sin x-\sin y|+\frac{e^{-\cos^2 t}}{\sqrt{64+t}}|\tan^{-1}x-\tan^{-1}y| \\ &\le \frac{1}{7}|x-y|,\end{aligned}$$

then (H_1) is satisfied with $L=\frac{1}{7}$.

A simple computation givens $\Delta_1 \le 0.24428$ (or $\Delta_1 \ge 0.45432$), $\Delta_2 = 2.98095$, $\Delta_3 \approx 1.15446$, $\Theta = 4.63847$, $L\Theta \approx 0.66264 < 1$, Hence by Theorem 3.1, (1.1) has a unique solution on $[0,1]$.

4.2

$$\begin{cases} {}^cD_{q,\omega}^{\frac{4}{3}}x(t)=\frac{1}{6}e^{-t^2}\ln(1+|x|),\ t\in[0,1], \\ {}^cD_{q,\omega}^{\frac{4}{5}}x(\frac{1}{11})=\frac{2}{7}D_{q,\omega}^{\frac{1}{3}}x(\frac{1}{8})+\frac{11}{12}{}^cD_{q,\omega}^{\frac{1}{3}}x(\frac{2}{9}), \\ x(1)=\frac{1}{5}\int_0^{\frac{1}{4}}x(s)d_{q,\omega}s+\frac{3}{2}\int_0^{\frac{2}{3}}x(s)d_{q,\omega}s+\frac{1}{2}x(\frac{1}{8})+x(\frac{8}{9}), \end{cases} \tag{4.2}$$

where $w=\frac{4}{3}$, $q=\frac{1}{2}$, $v=\frac{1}{3}$, $\sigma=\frac{4}{5}$, $\eta_1=\frac{1}{4}$, $\eta_2=\frac{2}{3}$, $\xi=\frac{1}{11}$, $\alpha_1=\frac{2}{7}$, $\alpha_2=\frac{11}{12}$, $\beta_1=\frac{1}{5}$, $\beta_2=\frac{3}{2}$, $\gamma_1=\frac{1}{2}$, $\gamma_2=1$, $f(t,x)=\frac{1}{6}e^{-t^2}\ln(1+|x|)$, $0.506784 \le \Delta_1 \le 0.55299$, $\Delta_2 = 2.55$, $\Delta_3 \approx -0.36831$, let $g(t)=\frac{1}{6}e^{-t^2}$, we obtain $\phi = 0.89077$ .Clearly,

$$\begin{aligned}|f(t,x)-f(t,y)| &\le g(t)\big(\ln(1+|x|)-\ln(1+|y|)\big) \\ &\le g(t)\phi^{-1}\ln(1+|x-y|).\end{aligned}$$

Hence, by Theorem 3.2, the nonlocal boundary value problem (4.2) has a unique solution on [0,1].

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