International Journal of Latest Engineering and Management Research (IJLEMR)
ISSN: 2455-4847
www.ijlemr.com || Volume 04 - Issue 03 || March 2019 || PP. 67-71

Global Existence for Benjamin-Bona-Mahony's Equation in Weighted Spaces and Applications

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Abstract: We prove the global existence for the Benjamin-Bona-Mahony's equation (BBM) in a weighted Sobolev spaces. We used the theory of semigroups of linear operators so as the interpolation space and Sobolev's embedding the orem for obtaining a mild solution and some results of functional analysis.First of all It has been proved thelocal existence of solution for the Benjamin-Bona-Mahony's equation (BBM) in a weighted Sobolev space and so this one it follows the result of global existence.Theglobal existence of solution for the Benjamin-Bona-Mahony's equation (BBM) in a weighted Sobolev space follows by use of semigroups of linear operators.

Keywords: Global solution,weighted Sobolev space, DNA breathing.

I. Introduction

The behavior of the natural phenomena can be analyzed with the rate of changes of the dependent variable. These rates can be expressed in mathematical relations using derivatives. For instance, in biology, the vibrational problems, DNA breathing and wave propagations; in epidemiology , the model SIR and so on.
We consider the initial value problem for the regularized long-wave equation of Peregrine[12] and Benjamin-Bona-Mahony[1]:

$$\partial_t u + \partial_x u + u\partial_x u - \partial_x^2 \partial_t u = 0 \quad (1)$$

For all $x \in \mathbb{R}$, $t > 0$

We are concerned to the global existence for the Benjamin-Bona Mahony's equation in a weighted Sobolev space (1) .When the initial data is given in a suitable weighted Sobolev space. In this case the weight is of polynomial type. The reason is the following: Suppose that we have a global solution $u(x,t)$ of (1) with initial data in the usual Sobolev space $H^m(\mathbb{R})$. One knows that the equation (1) has traveling wave solution of the type [12]:

$$u(x,t) = 3(c-1)\operatorname{sech}^2\left(\frac{x-ct}{2}\sqrt{\frac{c-1}{c}}\right)$$

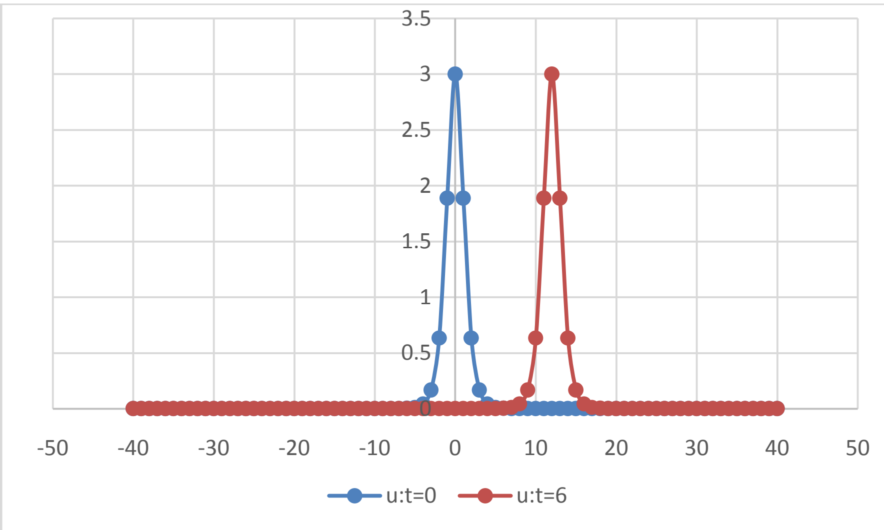

Fig.1: Solution of the equation BBM using the parameterc=2.

The traveling wave is graphed in Fig.1 for the parameter c=2 in the domain [-40.40].

Furthermore we know that the solutions are stable. One of the problems about these solutions is that there exists one impossibility of estimating the difference:

$\|u(.,t) - u_c(.)\|_m$ as $t \to \infty$ where $u_c(x) = 3(c-1)\text{sech}^2\left(\frac{x}{2}\sqrt{\frac{c-1}{c}}\right)$

These problems has been treated by an alternative method by Miller, J. and Weinstein, M. [9] using weighted Sobolev space with exponential type weights. Technical reason makes the procedure more simple in this case, a conjecture in [9] is that similar results should be valid with weights of polynomial type and that is the main result of our present research.

Global existence for the equation (1) with initial data in the usual Sobolev space was studied by many authors (cf.[1],[2],[3],[6],[10]). Similar equations in this context of weighted space was studied in (cf [5],[7]).Similar solutions were obtained for the DNA breathing in the Peyrard-Bishop model of DNA. The traveling wave is graphed in Fig.2.[14]

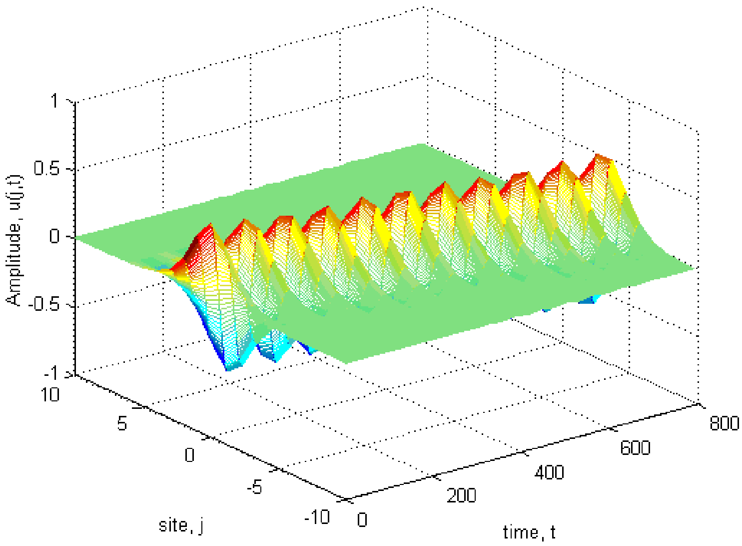

Figure2 Spatial mobile breather configuration (μ=0.35, wb=0.8) for the SPB model

II. Methodology

We introduce the weighted Sobolev space and stablish some properties which are essential tools used in subsequent sections. We will consider the initial value problem for the equation (1) with the initial condition in a weighted Sobolev space.

Notation and preliminaries

Let X a Hilbert space and $m \in \mathbb{N}$, $T > 0$ we denote by $C^m([0,T];X)$ the space of vector. Valued functions $u:[0,T] \to X$ m-times continuously differentiable in$[0,T]$ with the usual norm:

$$\mu(x) = (1 + |x|^2)^{1/2}$$

Forevery$s \in \mathbb{R}$, the Sobolev spaces $H^s(\mathbb{R})$ are introduced as usual with the norm
$\|u\|_s = |\mu^s \hat{u}|_2$where the $|.|$ denotes the standard norm of $L^2(\mathbb{R})$ and $\hat{u}$ is the Fourier transform of u . We also denote by

$$\Lambda^s = (I - \partial_x^2)^{s/2}$$

The Bessel potential of order s.

If $r \in \mathbb{R}$ we denote by M_r the multiplication operator with the function $\mu^r(x)$. While if $r, s \in \mathbb{R}$ we denote by $H_r^s(\mathbb{R}) = H^s(\mathbb{R}, \mu^r(x)dx)$ the completion of Schwarz space $\mathcal{S}(\mathbb{R})$ in the norm

$$|u|_{r,s} = |M_r \Lambda^s u|_2$$

$H_r^s(\mathbb{R})$is a Hilbert space with the inner product
$(u,v)_{r,s} = (M_r\Lambda^s u, M_r\Lambda^s v)_2$and its dual is defined by $\left(H_r^s(\mathbb{R})\right)' = H_{-r}^{-s}(\mathbb{R})$. When $r = 0$, the space H_0^s coincide with the usual norm Sobolev space. The case $r = s = 0$ we put
$H_0^0(\mathbb{R}) = L^2(\mathbb{R})$.
We introduce the Hilbert space $\mathcal{S}_r^s(\mathbb{R})$ defined by

$$\mathcal{S}_r^s(\mathbb{R}) = H_r^0(\mathbb{R}) \cap H_0^s(\mathbb{R})$$

For every $r, s \in \mathbb{R}$ with the inner product

$$(((u,v)))_{r,s} = (u,v)_{r,0} + (u,v)_{0,s}$$

We give some properties
i)Forevey$r_1, s_1, r_2, s_2 \in \mathbb{R}$ and $\theta \in (0,1)$

$$\left[H_{r_1}^{s_1}, H_{r_2}^{s_2}\right]_\theta = H_{(1-\theta)r_1+\theta r_2}^{(1-\theta)s_1+\theta s_2}$$

Where $[X,Y]_\theta$ is the interpolation space of X and Y.
ii) For every$r, s \in \mathbb{R}$there exists c_1, c_2 positive constants such that

$$c_1|u|_{r,s} \le |\mathcal{F}^{-1}(M_s\mathcal{F}(M_r u))|_2 \le c_2|u|_{r,s}$$

For every $s \in H_r^s(\mathbb{R})$
iii)For every $r, s \in \mathbb{R}$, $\mathcal{F}(H_r^s(\mathbb{R})) = H_s^r(\mathbb{R})$
iv) If$r_1 \ge r_2, s_1 \ge s_2$ then
$H_{r_1}^{s_1} \subset H_{r_2}^{s_2}$, $\mathcal{S}_{r_1}^{s_1} \subset \mathcal{S}_{r_2}^{s_2}$with continuous injection.
v) For every $r, s \in \mathbb{R}$ and $\theta \in (0,1)$

$$\mathcal{S}_r^s(\mathbb{R}) \subset [H_r^0(\mathbb{R}), H_0^s(\mathbb{R})]_\theta = H_{(1-\theta)r}^{\theta s}$$

vi) If$r_1, s_1, r_2, s_2, r, s \in \mathbb{R}$ and $u \in H_{r_1}^{s_1}(\mathbb{R}), v \in H_{r_2}^{s_2}(\mathbb{R})$ then $uv \in H_r^s(\mathbb{R})$ whenever

$$s_1, s_2 \ge s, \qquad s_1 + s_2 - s \ge 1/2$$

Moreover there exists a constant $c > 0$ such that:

$$|uv|_{r,s} \le c|u|_{r_1,s_1}|u|_{r_2,s_2}$$

For some other details one find in (cf.[4],[14].

III. Result and Discussion

Our main theorem on global existence

Theorem 1. Let $r \ge 0$ if $u_0 \in H^4(\mathbb{R}) \cap H_r^2(\mathbb{R})$
then the initial value problem

$$\begin{cases} \partial_t u + \partial_x u + u\partial_x u - \partial_x^2 \partial_t u = 0 \,, x \in \mathbb{R}, t > 0 \\ \\ u(x,0) = u_0(x) \quad , \qquad x \in \mathbb{R} \end{cases}$$

(2)
Has a unique strong global solution

$$u \in C^0([0,\infty); H^4(\mathbb{R}) \cap H_r^2(\mathbb{R})) \cap C^1([0,\infty); H_r^2(\mathbb{R}))$$

Before proceeding to the proof, we establish some preliminary results. We know that (2) is equivalent to the following initial-value problem:

$$\begin{cases} \partial_t u = Bu + Bfu \quad , x \in \mathbb{R}, t > 0 \\ \\ u(x,0) = u_0(x) \quad , \qquad x \in \mathbb{R} \end{cases}$$

Where $B = -\Lambda^{-2}\partial_x$, $f(s) = s^2/2$

Lemma1. $B: H_r^2(\mathbb{R}) \to H_r^2(\mathbb{R})$is a linear bounded operator.
Proof. In fact, for $u \in H_r^2(\mathbb{R})$ we have

$$|Bu|_{r,2}^2 = \left|M_r\Lambda^2\Lambda^{-2}\partial_y u\right|_2^2 = \left|M_r\partial_y u\right|_2^2 =$$

$$\int_{\mathbb{R}} (1+|y|^2)^r \,\partial_y u \partial_y u dy = -$$

$$-\int_{\mathbb{R}} u\partial_y((1+|y|^2)^r\,\partial_y u)dy = -$$

$$-\int_{\mathbb{R}} 2r(1+|y|^2)^{r-1}\,yu\partial_y udy -$$
$$-\int_{\mathbb{R}} (1+|y|^2)^r u_{yy}\,udy$$

Since that $y^2 \leq (1+|y|^2)^2$

It follows from the Cauchy-Schwarz's inequality

$$|Bu|_{r,2}^2 \leq$$
$$2r\left|M_r\partial_y u\right|_2 . |M_r u|_2 + |M_r u|_2\left|M_r\partial_y^2 u\right|_2$$

On the other hand, an elementary calculation basedon the identity

$$\left|\Lambda^2 u\right|^2 = \left|\left(I-\partial_y^2\right)u\right|^2 = |u|^2 +$$
$$+\left|\partial_y^2 u\right|^2 - 2u\partial_y^2 u$$

Shows that

$$\left|\partial_y^2 u\right|_{r,0}^2 \leq c|u|_{r,0}^2 + |u|_{r,2}^2$$

And so, by applying the embedding $H_r^2 \subset H_r^0$

It follows that

$$|Bu|_{r,2}^2 = \left|\partial_y^2 u\right|_{r,0}^2 \leq c|u|_{r,2}^2 + |u|_{r,2}^2$$

Hence

$$|Bu|_{r,2} \leq c|u|_{r,2}$$

Proposition1. (local existence)Given $u_0 \in H_r^2(\mathbb{R})$ there exists $T \in (0,\infty)$ and an unique mild solution u of (1) in $\mathbb{R} \times (0,T)$ with initial data u_0. Here a mild solution is an element of $C([0,T];H_r^2(\mathbb{R}))$ satisfying

$$u(t) = e^{Bt}u_0 + \int_0^t e^{B(t-\sigma)}B\left(\frac{u(\sigma)^2}{2}\right)d\sigma$$

We use Lemma1 for concluding that B is the infinitesimal generator of a uniformly continuous semigroup of operators on $H_r^2(\mathbb{R})$. The conclusion follows using standard arguments given in Pazy[11].

1. Proof of Theorem 1

Note that Bf is differentiable, $u(x,t)$ is a classical solution of (1) on any t −interval where it exists. Furthermore

$$\partial_t u = Bu + Bfu \in C([0,T];H_r^2(\mathbb{R}))$$

For every $s \geq 0$ the existence of a globalH^s −valued solution for (1) has already been proved (cf.[1],[2]). In addition such solution belongs to $C^0([0,\infty);H^s(\mathbb{R}))$.

We claim that for every $T > 0$, there exists some constant$M_T > 0$ such that

$$\sup_{0\leq t\leq T}|u(.,t)|_{r,2} \leq M_T$$

In fact, for every $u_0 \in H_r^2(\mathbb{R})$ by the density of $\mathcal{S}_{2r}^4(\mathbb{R})$ in $H_r^2(\mathbb{R})$ (cf.[14])there exists

$(u_{0n}) \in \mathcal{S}_{2r}^4(\mathbb{R})$such that

$u_{0n} \to u_0$strong in $H_r^2(\mathbb{R})$

Now consider $u_n = u_n(x,t)$ be the solution of initial value problem.

$$\begin{cases}\partial_t u_n + \partial_x u_n + u\partial_x u_n - \partial_x^2\partial_t u_n = 0\,, x \in \mathbb{R}, t > 0 \\ \\ u_n(x,0) = u_{0n}(x) \quad , \qquad x \in \mathbb{R}\end{cases}$$

Then the solution u_n exists globally in$C([0,\infty);\mathcal{S}_{2r}^4(\mathbb{R}))$, (cf.[13]). In particular given any $T > 0$ there exists a constant M_T(independentof n) such that

$$\left|\left|\left|u_n\right|\right|\right|_{2r,4} \leq M_T$$

And (u_n) converges to the solution $u = u(x,t)$ of the problem (2). Since we have the embedding injection $\mathcal{S}_{2r}^4(\mathbb{R}) \subset H_r^2(\mathbb{R})$ it follows that

$$|u_n(.,t)|_{r,2} \leq C\left|\left|\left|u_n\right|\right|\right|_{2r,4} \leq M_T$$

Finally, we claim that

$$|u(.,t)|_{r,2} \leq \lim_{n\to\infty} inf|u_n(.,t)|_{r,2} \leq M_T$$

This completes the proof.
Actually we have new types of wave solutions of the BBM equation or regularized long-wave equation (RLW) : v using the two variable (G'/G,1/G)- expansion method [6] for the study of the $\|u(.,t) - v(.)\|_m$ as $t \to \infty$.

IV. Conclusion

The result of theorem 1 still holds for the case $r = 0$, see Weinstein [9]. Actually we have proved the global existence of solution for the equation (1) and so it was possible its generalized for weights of polynomial type As we saw, the proof of theorem 1 was possible even if whether or not theinitial dataare small. On the other hand up to now, it is unknown whether or notthere exists the global solution for the other more general weighted.

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