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The Skelton graph of vector spaces

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Abstract: In this paper, we introduce a new graph structure, called skeleton graph on finite dimensional vector spaces. In this chapter, we study connectedness, complete, tree, and Eulerian properties of the skeleton graph. Moreover, we characterize all finite dimensional vector space $\mathbb{V}$ for which the skeleton graph is toroidal.
Keywords: Book thickness, non-cyclic graph, coprime graph, join graph of subgroups of a group

1. Introduction

The first instances of associating graphs with various algebraic structures is due to Beck [5] who introduced the idea of zero divisor graph of a commutative ring with unity. Though his key goal was to address the issue of colouring, this initiated the formal study of exposing the relationship between algebra and graph theory and at advancing applications of one to the other. Till then, a lot of research, e.g., [15, 2, 3, 1, 8, 6, 7, 4] has been done in connecting graph structures to various algebraic objects. Intersection graphs associated with subspaces of vector spaces were studied in [14, 18]. Recently, some other types of graphs associated with vectors in finite dimensional vector spaces were studied in [9, 11, 13].

Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$ with $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ as a basis and θ as the null vector. Then any vector $v \in \mathbb{V}$ can be expressed uniquely as a linear combination of the form $v = a_1\alpha_1 + a_2\alpha_2 + + a_n\alpha_n$. In [10], A. Das defined *skeleton* of v with respect to B, as

$$S_B(v) = \{\alpha_i : a_i \,\square\, 0, i \in \{1, 2, \ldots, n\}\}.$$

He defined a graph(Skeleton Union Graph) $\Gamma_{\mathbb{F}}(V_{\mathbb{B}}) = (V, E)$ (or simply $\Gamma(\mathbb{V})$ or $\Gamma(\mathbb{V}_B)$ as follows:

$V = \mathbb{V} \setminus \{\theta\}$ and for $v_1, v_2 \in V$, $v_1 \sim v_2$ or $(v_1, v_2) \in E$ if and only if $S_B(v_1) \cup S_B(v_1) = B$.

Throughout this chapter, vector spaces $\mathbb{V}$ are finite dimensional over a field $\mathbb{F}$ and $n = \dim_F(\mathbb{V})$. Unless otherwise mentioned, we take the basis on which the graph is constructed as $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.

2. Basics Properties of Skeleton Graphs

In this section, we introduce a new graph structure, called skeleton graph on finite dimensional vector spaces. Further, we study connectedness, complete, tree, and Eulerian properties of the skeleton graph.

Definition 2.1. The skeleton graph $\Gamma'(\mathbb{V}_B)$ of $\mathbb{V}$ with respect to B is a simple graph with vertex set $V = \{v \in \mathbb{V} \setminus\{\theta\} : S(v) \subset B\}$ and two vertices v_1 and v_2 are adjacent in $\Gamma(\mathbb{V}_B)$ if and only if $S_B(v_1) \cup S_B(v_2) = B$.

Example 2.2. Let $\mathbb{V} = \mathbb{F}_4$ be a vector space over $\mathbb{F}_2$. Then $V\left(\Gamma'(\mathbb{V}_B)\right) = \{\alpha_1, \alpha_2\}$ and so $\Gamma'(\mathbb{V}_B)\gamma \cong K_2$.

Theorem 2.3. *Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$. Then $\Gamma'(\mathbb{V}_B)$ is empty graph if and only if* $\dim(\mathbb{V}_B) = 1$

Proof. Assume that $\dim \mathbb{V} = 1$. Let $\alpha, \beta \in \mathbb{V}^*$. Then $S(\alpha) = \{\alpha_1\} = S(\beta) = B$ and so $V(\Gamma'(\mathbb{V}_B))$ is empty. Hence $\Gamma'(\mathbb{V}_B)$ is empty graph.
Conversely, assume that $\Gamma'(\mathbb{V}_B)$ is an empty graph. Suppose that $\dim \mathbb{V} > 1$.
Then there exist $\alpha_1 \in \mathbb{V}^*$ and
$\alpha_2 + \alpha_3 + \cdots\cdots + \alpha_n \in \mathbb{V}^*$ such that $S(\alpha_1)$, $S(\alpha_2 + \alpha_3 + \cdots\cdots + \alpha_n) \subset B$ and $S(\alpha_1) \cup S(\alpha_2 + \cdots\cdots + \alpha_n) = B$. Hence α_1 and $\alpha_2 + \cdots\cdots + \alpha_n$ are adjacent in $\Gamma'(\mathbb{V}_B)$, a contradiction. Hence $\dim(\mathbb{V}_B) = 1$.
In view of Theorem 2.3, rest of this chapter, we assume that $\mathbb{V}$ is a vector space of dimension greater than 1.

Theorem 2.4. Let $\mathbb{V}$ be a finite dimensional vector space over a finite field $\mathbb{F}$. Then $\Gamma'(\mathbb{V}_B)$ is connected and $1 < \operatorname{diam}(\Gamma'(\mathbb{V}_B)) \leq 3$.

Proof. Let $\alpha, \beta \in \mathbb{V}^*$. If α and β are adjacent in $\Gamma'(\mathbb{V}_B)$, then d (α, β) = 1.
Suppose α and β are not adjacent. Then S (α) ∪ S (β) □ B. Let U = B − S (α).
If S (α) = S (β), then there exist γ in $\mathbb{V}^*$ such that S (γ) = B −S (α) and so α −γ –β is a path in $\Gamma'(\mathbb{V}_B)$.
If S (α) ⊂ S (β), then there exist γ in $\mathbb{V}^*$ such that S (γ) = B − S (β) and so α− γ − β is a path in $\Gamma'(\mathbb{V}_B)$.
If S (α) ∩ S (β) = □, then there exist γ_1, γ_2 in $\mathbb{V}^*$ such that S (γ_1) = B − S (α) and S (γ_2) = B − S (β) and so $\alpha - \gamma_1 - \gamma_2 - \beta$ is a path in $\Gamma'(\mathbb{V}_B)$.
If S (α) ∩S (β) = □, then there exist γ in $\mathbb{V}^*$ such that S (γ) = B −(S (α) ∩ S (β)) and so α − γ − β is a path in $\Gamma'(\mathbb{V}_B)$. Hence in any case d (α, β) ≥ 2 and so diam ($\Gamma'(\mathbb{V}_B)$) ≤ 3.

Theorem 2.5. Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$. Then $\Gamma'(\mathbb{V}_B)$ is complete bipartite if and only if dim ($\mathbb{V}$) = 2.

Proof. Suppose $\Gamma'(\mathbb{V}_B)$ is complete bipartite. If dim ($\mathbb{V}$) ≥ 3, then there exist $u_i = \alpha_1 + \alpha_2 + \cdots + \alpha_{i-1} + \alpha_{i+1} + \cdots + \alpha_n$ in $\mathbb{V}^*$ such that $u_1 - u_2 - u_3 - u_1$ is a 3-cycle in $\Gamma'(\mathbb{V}_B)$, a contradiction. Hence dim ($\mathbb{V}$) = 2.
Conversely, suppose dim ($\mathbb{V}$) = 2. For any α in V ($\Gamma'(\mathbb{V}_B)$), $\alpha = u\alpha_1$ or $\alpha = v\alpha_2$, $S_1 = \{u\alpha_1 : u \text{ in } \mathbb{F}\}$ and $S_2 = \{v\alpha_2 : v \text{ in } \mathbb{F}\}$. Then every vertices in S_1 is adjacent to every vertices in S_2 and Hence $\Gamma'(\mathbb{V}_B)$ is complete bipartite.

Corollary 2.6. Let $\mathbb{V}$ be a finite dimensional vector space over a finite field $\mathbb{F}$. Then $\Gamma'(\mathbb{V}_B) \cong C_4$ if and only if dim ($\mathbb{V}$) = 2 and F $\mathbb{F} \cong \mathbb{Z}_3$

Proof. If $\Gamma'(\mathbb{V}_B) \cong C_4$ then by Theorem 2.5, dim ($\mathbb{V}$) = 2. If $|\mathbb{F}| = 2$, then $\Gamma'(\mathbb{V}_B) \cong K_2$, a contradiction.
If $|\mathbb{F}| \geq 4$, then $\Gamma'(\mathbb{V}_B) \cong \mathbb{K}_{3,3}$, a contradiction. Hence $\mathbb{F} \cong \mathbb{Z}_3$

Corollary 2.7. Let $\mathbb{V}$ be a finite dimensional vector space over a finite field $\mathbb{F}$. Then $\Gamma(\mathbb{V}_B)$ is tree if and only if dim $(\mathbb{V}) = 2$ and $\mathbb{F} \cong \mathbb{Z}_3$

Corollary 2.8. Let $\mathbb{V}$ be a finite dimensional vector space over a finite field $\mathbb{F}$. Then $\Gamma'(\mathbb{V}_B)$ is complete if and only if dim $(\mathbb{V}) = 2$ and $\mathbb{F} \cong \mathbb{Z}_2$

Proof. Suppose $\Gamma'(\mathbb{V}_B)$ is complete. Suppose dim $(\mathbb{V}) \geq 3$. Then α_1 and α_2 are non-adjacent in $\Gamma'(\mathbb{V}_B)$, a contradiction. Hence dim ($\mathbb{V}$) = 2. If $|\mathbb{F}| \geq 3$, then C_4 is a subgraph of $\Gamma'(\mathbb{V}_B)$, a condradiction. Hence $\mathbb{F} \cong \mathbb{Z}_2$

Theorem 2.9. Let V be a finite dimensional vector space over a field $\mathbb{F}$.
Then $\delta(\Gamma'(\mathbb{V}_B))$= deg($\alpha_i$) for all i = 1 to n.

Proof. Let B = $\{\alpha_1, \ldots, \alpha_n\}$ be a basis for a vector space $\mathbb{V}$.
Let $S_i = \{\lambda_1\alpha_1 + \cdots + \lambda_{i-1}\alpha_{i-1} + \lambda_{i+1}\alpha_{i+1} + \cdots + \lambda_n\alpha_n : \lambda_j \in \mathbb{F}^*\}$ for i = 1 to n.
Then $|S_i| = (\mathbb{F} - 1)^{n-1}$. For each x in S_i,
S (x) = $\{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n\}$.
Clearly S (x) ∪ S (α_i) = B and so α_i is adjacent to x for all x in S_i. Hence deg (α_i) = $|S_i|$. Let y □ α_i for all i. Then there exist k, l such that y = ...+ $\lambda_k \alpha_k + \lambda_l \alpha_l$ + …. in V($\Gamma'(\mathbb{V}_B)$) and S(y) □ B. Therefore y is adjacent to all elements of S_k and y is adjacent to all elements of S_l . Hence deg (y) > deg (α_i) and so $\delta(\Gamma'(\mathbb{V}_B))$ = deg (α_i) for all i.

Theorem 2.10. Let $\mathbb{V}$ be a finite dimensional vector space over a field $\mathbb{F}$ and $|V(\Gamma'(\mathbb{V}_B))| \geq 3$.
Then $\Gamma'(\mathbb{V}_B)$ is Eulerian if and only if $|\mathbb{F}|$ is odd.

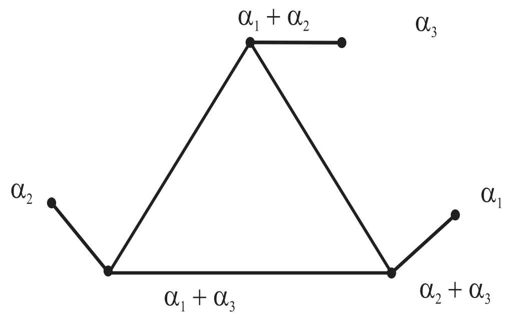
Proof. Suppose $\Gamma'(\mathbb{V}_B)$ is Eulerian. If $|\mathbb{F}|$ is even, then deg (α_i) = $(|\mathbb{F}| - 1)^{n-1}$ is odd, a contradiction.
Hence $|\mathbb{F}|$ is odd.
Conversely, suppose $|\mathbb{F}|$ = q is odd.
Then deg $(\alpha_1 + \cdots + \alpha_k) = \{\lambda_1\alpha_1 ++ \lambda_{k+1}\alpha_{k+1} + \cdots + \lambda_n\alpha_n : \lambda_i$ □ $0\}$
for i=1 to n. Then deg $(\alpha_1 + \cdots + \alpha_k) = q^k (q-1)^{n-k}$ and so deg $(\alpha_1 + \cdots + \alpha_k) = q^{n-1}(q-1)$ is even.
Hence $\Gamma'(\mathbb{V}_B)$is Eulerian.

3. Topological properties of $\Gamma(\mathbb{V}_B)$

Theorem 3.1. *Let* V = $\mathbb{F}_p{}^n$ be a vector space over a field $\mathbb{F}_p$. Then $\Gamma'(\mathbb{V}_B)$ is planar if and only if the one of the following is holds: p = 2 and n ≤ 3 (or) p = 3 and n = 2.

Proof. If n = 1, then by Theorem 2.3, $\Gamma'(\mathbb{V}_B)$ is an empty graph. Hence we assume that n ≥ 2.
Suppose p ≥ 5. Then $K_{3,3}$ is a subgraph of $\Gamma'(\mathbb{V}_B)$, a contradiction. Hence p ≤ 3.
Suppose p ≥ 3. If n ≥ 3, then $K_{3,3}$ is subgraph of $\Gamma'(\mathbb{V}_B)$, a contradiction.
Hence p = 2. Suppose p=2, If n ≥ 4, $K_{3,3}$ is subgraph of $\Gamma(\mathbb{V}_B)$. Since $\Gamma(\mathbb{V}_B)$ is non-planar, a contradiction. Hence n ≤ 3.
Conversely if, Assume that if p=2 and n ≤ 3 and p=3 and n=2.

$\Gamma(\mathbb{V}_B)$: p = 2, n = 2



$\Gamma(\mathbb{V}_B)$: p = 2, n = 3

$\Gamma(\mathbb{V}_B)$: p = 3, n = 2

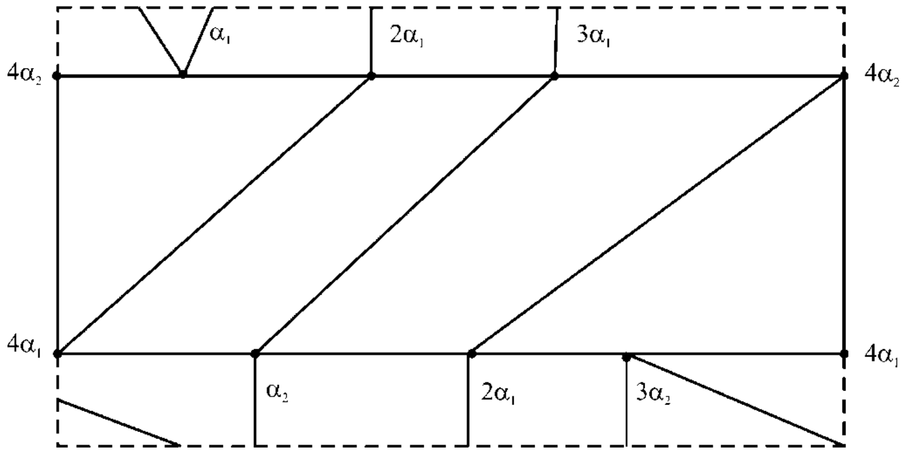
fig 6.1 Planar Embedding of $\Gamma(\mathbb{V}_B)$

Theorem 3.2. Let V = $\mathbb{F}_p{}^n$ be a vector space over a field $\mathbb{F}_p$. Then $\Gamma(\mathbb{V}_B)$ is outer-planar if and only if the one of the following is true: p = 2 and n ≤ 3 (or) p=3 and n=2.

Proof. The proof follows by above theorem.

Theorem 3.3. Let V=$\mathbb{F}_p{}^n$ be a vector space over a field $\mathbb{F}_p$. Then $\Gamma(\mathbb{V}_B)$ is toroidal if and only if $\Gamma(\mathbb{V}_B)$ if p=5 and n=2.

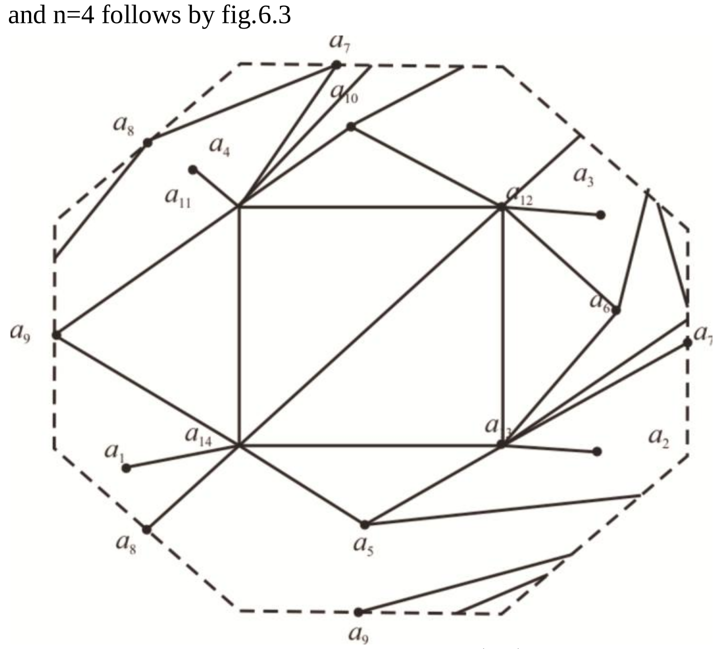
Proof. Assume that $\Gamma(\mathbb{V}_B)$ is toroidal. Suppose that p ≥ 5. If n ≥ 3, $|V(\Gamma(\mathbb{V}_B))| \geq 14$, $|E(\Gamma(\mathbb{V}_B))| \geq 25$ and $gr(\Gamma(\mathbb{V}_B)) = 3$ then by theorem that $\gamma(\Gamma(\mathbb{V}_B)) > 1$, a contradiction. Hence p=5 and n=2.
Suppose p=3. If n ≥ 3, $|V(\Gamma(\mathbb{V}_B))| \geq 18$, $|E(\Gamma(\mathbb{V}_B))| \geq 72$ and $gr(\Gamma(\mathbb{V}_B)) = 3$ then by theorem that $\gamma(\Gamma(\mathbb{V}_B)) > 1$, a contradiction.
Suppose p=2. If n ≥ 4, $|V(\Gamma(\mathbb{V}_B))| \geq 14$, $|E(\Gamma(\mathbb{V}_B))| \geq 25$ and $gr(\Gamma(\mathbb{V}_B)) = 3$ then by theorem that $\gamma(\Gamma(\mathbb{V}_B)) > 1$, a contradiction. Hence $\Gamma(\mathbb{V}_B)$ is toroidal if p=5 and n=2.
Conversely, if p = 5 and n = 2, then follows by fig. 6.2



$\Gamma(\mathbb{V}_B)$:p=5 n=2
fig 6.2. Embedding of $\Gamma(\mathbb{V}_B)$ *in* $\mathbb{S}_1$

Theorem 3.4. Let V=$\mathbb{F}_p{}^n$ be a vector space over a field $\mathbb{F}_p$. Then $\Gamma(\mathbb{V}_B)$ is genus two if and only if $\Gamma(\mathbb{V}_B)$ is isomorphic if p=2 and n=4.

Proof. Assume that $\Gamma(\mathbb{V}_B)$ is genus two.
Suppose p ≥ 3
If n ≥ 2, $|V(\Gamma(\mathbb{V}_B))| \geq 18$, $|E(\Gamma(\mathbb{V}_B))| \geq 72$ and gr($\Gamma(\mathbb{V}_B)$) = 3 then by theorem that $\gamma(\Gamma(\mathbb{V}_B)) > 1$, a contradiction. Hence p = 2
Suppose p = 2 ,If n ≥ 5, $|V(\Gamma(\mathbb{V}_B))| \geq 18$, $|E(\Gamma(\mathbb{V}_B))| \geq 72$ and gr($\Gamma(\mathbb{V}_B)$) = 3 then by theorem that $\gamma(\Gamma(\mathbb{V}_B)) > 1$, a contradiction. Hence n=4. Hence p=2 and n=4 be a genus two graph.
Conversely, if p=2 and n=4 follows by fig.6.3



$\Gamma(\mathbb{V}_B)$:p=2 n=4
fig 6.3. Embedding of $\Gamma(\mathbb{V}_B)$ *in* $\mathbb{S}_2$

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