International Journal of Latest Engineering and Management Research (IJLEMR)
ISSN: 2455-4847
www.ijlemr.com || Volume 04 - Issue 10 || October 2019 || PP. 14-19

Reduced trees and the algebra D[X] on Pre A*-algebra

Vijayabarathi.S
Assistant Professor, Department of Mathematics.
Sri Chandra sekharendra Saraswathi Viswa Mahavidyalaya, Kanchipuram

Abstract: In this paper, reduced trees has been defined on Pre A*-algebra. There is an homomorphism from D[X] onto F[X], where F[x] is a function in A(A^X) supported with 4 lemmas and two theorems with proof. $\Gamma_x(p,q)$ be a binary tree with root labeled x, left branch p and right branch q.An element of the free Pre A*-algebra on a set X will be an equivalence class of such trees, where p and q either have been one of 1,0 (or 2) In order to define these precisely , we will use a ternary operation Γ ,elements of X and 1,0(or 2).
Keywords: Reduced trees, free Pre A*-algebra,D[X] and F[X]

Introduction:

In 1994, P.KoteswaraRao [1] first introduced the concept of A*-algebra $\left(A,\wedge,\vee,*,(-)^{\sim}\,(-)_{\pi},0,1,2\right)$ In 2000, J.Venkateswara Rao[2] introduced the concept Pre A*-algebra $(A,\wedge,\vee,(-)^{\sim})$ analogous to C-algebra as a reduct of A*- algebra, K.SrinivasaRao[3] describes the concept of Pre A*-Algebra as a Poset and established necessary conditions for a poset to become a lattice with respect to meet and as well as join. J.Venkateswara Rao [4] analyze the properties of PreA*-function.He defined implicants of Pre A*-algebra function[5].In this paper, Positive PreA*-algebra function is defined with two theorems

1. Pre A*-algebra

In this section, we concentrate on the algebraic structure of Pre A*- algebra and C-algebra. We recalled some fundamental results which are also used in the later text.

Definition1.1[3]:

An algebra $(A,\wedge,\vee,(-)^{\sim})$ where A is non-empty set with 1, $\wedge$, $\vee$ are binary operations and $(-)^{\sim}$ is a unary operation satisfying

(a) $x^{\sim\sim}=x, \quad \forall x\in A$
(b) $x\wedge x = x, \quad \forall x\in A$
(c) $x\wedge y = y\wedge x, \quad \forall x,y\in A$
(d) $(x\wedge y)^{\sim} = x^{\sim}\vee y^{\sim}, \quad \forall x,y\in A$
(e) $x\wedge(y\wedge z) = (x\wedge y)\wedge z, \quad \forall x,y,z\in A$
(f) $x\wedge(y\vee z) = (x\wedge y)\vee(x\wedge z), \quad \forall x,y,z\in A$
(g) $x\wedge y = x\wedge(x^{\sim}\vee y), \quad \forall x,y\in A$.

is called Pre A*-algebra.

Example 1:2 = {0, 1, 2} with operations $\wedge,\vee,(-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^{\sim}$
0	1
1	0
2	2

Lemma1.3[3]: Every Pre A*-algebra with 1 satisfies the following laws

$x\vee 1 = x\vee x^{\sim}$
$x\wedge 0 = x\wedge x^{\sim}$

Lemma1.4[3]: Every Pre A*-algebra with 1 satisfies the following laws.

$x \wedge (x^{\sim} \vee x) = x \vee (x^{\sim} \wedge x) = x$

$(x \vee x^{\sim}) \wedge y = (x \wedge y) \vee (x^{\sim} \wedge y)$

$(x \vee y) \wedge z = (x \wedge z) \vee (x^{\sim} \wedge y \wedge z)$

Definition1.5[3]: Let A be a Pre A*-algebra. An element $x \in A$ is called central element of A if $x \vee x^{\sim} = 1$ and the set$\{ x \in A / x \vee x^{\sim} = 1 \}$ of all central elements of A is called the centre of A and it is denoted by $B(A)$.

Theorem1.6[3]: Let A be a Pre A*-algebra with 1, then $B(A)$ is a Boolean algebra with the induced operations $\wedge, \vee, (-)^{\sim}$

Definition1.7[4]: Let $(A_1, \vee, \wedge, (-)^{\sim})$ and $(A_2, \vee, \wedge, (-)^{\sim})$ be a two Pre A*- algebras. A mapping $f : A_1 \to A_2$ is called an Pre A*-homomorphism if

$f(a \wedge b) = f(a) \wedge f(b)$

$f(a \vee b) = f(a) \vee f(b)$

$f(a^{\sim}) = (f(a))^{\sim}$.

The homomorphism $f : A_1 \to A_2$ is onto,and then f is called epimorphism.

The homomorphism $f : A_1 \to A_2$ is one-one then f is called monomorphism

The homomorphism $f : A_1 \to A_2$ f is one-one and onto then f is called an isomorphism, and A_1, A_2 are isomorphic, denoted in symbols $A_1 \cong A_2$.

Definition1.8.[4]: Let A_1, A_2 be two Pre A*-algebras and $f : A_1 \to A_2$ be a homomorphism then the set $\{x \in A_1 / f(x) = 0\}$ is called the Kernel of f and it is denoted by Ker f.

Lemma1.9.[4]: Let A be a Pre A*-algebra with 1,0. Suppose that for every $x \in A - \{0,1\}$, $x \vee x^{\sim} \neq 1$ define $f : A \to \{0,1,2\}$ by $f(1) = 1$, $f(0) = 0$ and $f(x) = 2$ if $x \neq 0,1$ then f is a Pre A*- homomorphism.

Lemma1.10.[4]: Let A be a Pre A*-algebra with 1,0. Suppose that for every $x \in A - \{0,1\}$, $x \vee x^{\sim} \neq 1$ define $f : A \to \{0,1,2\}$ by $f(1) = 1$, $f(0) = 0$ and $f(x) = 2$ if $x \neq 0,1$ then f is a Pre A*-homomorphism.

Definition 1.11.[2]: A relation θ on a Pre A*-algebra A is called congruence relation if

i. θ is an equivalence relation

ii. θ is closed under $\wedge, \vee, (-)^{\sim}$

Definition1.12[6]: Let A be a Pre A*-algebra and S is a nonempty subset of A.We define S as a Pre A*-subalgebra if it satisfies

i. If $a \in S$ then $a^{\square} \in S$

ii. If $a, b \in S$ then $a \wedge b \in S$

2. Reduced trees and the algebra D[X]

Definition2.1 : Let X be a set . Reduced tree t on X and the variables $X(t) \subseteq X$ that they involve are define recursively as follows.

1. 1,0 (and 2) are reduced trees. If t is one of these trees , then $X(t) = \phi$
2. Let x∈X.If p and q are reduced tree on X and $x \notin X(p) \cup X(q)$ then $\Gamma_x(p,q)$ is a reduced tree on X and $X[\Gamma_x(p,q)] = \{x\} \cup X(p) \cup X(q)$.

D[X] denote the set of all reduced trees on X. D[X] can be viewed as subsets of the term algebra involving the ternary operation Γ, generators X and constants 1,0 (and 2)

Definition2.2: Let X be a set , let $x \in X$, M = {0,1 } and $t \in D[X]$.

Define $t(x \leftarrow M)$ recursively as follows

1. If t is one of 1,0 (or 2) then $t(x \leftarrow M) = t$
2. If $t = \Gamma_y(p,q)$ with $y \neq x$ *then* $t[x \leftarrow M] = \Gamma_y(p[x \leftarrow M], q[x \leftarrow M])$
3. $t = \Gamma_x(p,q)$ *then* $t[x \leftarrow 1] = p \,\&\, t[x \leftarrow 0] = q$

Lemma2.3: Let X be a set , $x, y \in X$ with $x \neq y$. Let $M, N \in \{0,1\}$ and $t \in D[X]$ .Then

a. $t(x \leftarrow M)$ is a reduced tree on X & $X[t(x \leftarrow M)] = X(t) - x$
b. If $x \notin X(t)$, *then* $t[x \leftarrow M] = t$
c. $t(x \leftarrow M)(y \leftarrow N) = t(y \leftarrow N)(x \leftarrow M)$

Proof: It can be proved by induction on t.

a. $t(x \leftarrow M)$ is a reduced tree on X If t is any one of 1,0 (or 2)

$$X[1(x \leftarrow M)] = X(1) - x$$
$$X[0(x \leftarrow M)] = X(0) - x$$
$$X[2(x \leftarrow M)] = X(2) - x$$

$\therefore$ t is a reduced tree on X and $X[t(x \leftarrow M)] = X(t) - x$

b. *If* $x \notin X(t) \Rightarrow x$ is not any one of 1,0 (or 2), $\Gamma_x(p,q)$.

T is any one of 1,0 (or 2), $\Gamma_x(p,q)$.

By the previous statement $X[t(x \leftarrow M)] = X(t) - x \Rightarrow x \notin X(t)$

$$X[t(x \leftarrow M)] = X(t)$$
$$t(x \leftarrow M) = t$$

c. M = {0,1},N={ 0,1 }

$$t(x \leftarrow M)(y \leftarrow N) = t$$
$$t(y \leftarrow N)(x \leftarrow M) = t$$
$$t(x \leftarrow M)(y \leftarrow N) = t(y \leftarrow N)(x \leftarrow M)$$

Whatever the combinations such as (1,1),(1,0),(0,1),(0,0) we get

$$t(x \leftarrow M)(y \leftarrow N) = t(y \leftarrow N)(x \leftarrow M)$$

Definition2.4 : Let X be a set and $t_1, t_2 \in D(X)$

a. If t_1 is one of 1,0 (or 2) then $t_1^{\sim}, t_1 \wedge t, t_1 \vee t$ are defined in accordance with

$2^{\sim} = 2 : 1 \wedge x = x : 1^{\sim} = 0$ and $a^{\sim}$ is the left zero for $\wedge$. $\quad a^{\sim} = a^{\sim} \wedge x$

b. If $t_1 = \Gamma_x(p,q)$ we use the following

$$\Gamma_x(p,q)^{\sim} = \Gamma_x(p^{\sim}, q^{\sim})$$
$$\Gamma_x(p,q) \wedge t = \Gamma_x[p \wedge t(x \leftarrow 1), q \wedge t(x \leftarrow 0)]$$

$\Gamma_x(p,q) \vee t = \Gamma_x[p \vee t(x \leftarrow 1), q \vee t(x \leftarrow 0)]$ the argument of Γ_x on the right hand side have been defined previously.

Lemma2.5: Let X be a set . Let $t_1, t \in D[X]$, $x \in X$ *and* $M = \{0,1\}$ .Then

a. $t_1^{\sim}, t_1 \wedge t$ and $t_1 \vee t$ all belong to D[X]
b. $X(t_1^{\sim}) = X(t_1)$ and both $X(t_1 \wedge t)$ & $X(t_1 \vee t)$ are subsets of $X(t_1) \cup X(t)$
c. The function $t \to t(x \leftarrow M)$ is $a^{\sim}, \wedge$ and $\vee$ homomorphism from D[X] to itself.

Proof:

a. $\because t_1, t \in D[X] \Rightarrow t_1, t$ is any one of 1,0 (or 2), $\Gamma_x(p,q)$

If t is 1 then $t^{\sim} = 0 \in D[X]$; ,if t is 0 then $t^{\sim} = 1 \in D[X]$; if t is 2 then $t^{\sim} = 2 \in D[X]$ and if t is $\Gamma_x(p,q)$ then $t^{\sim} = \Gamma_x(p^{\sim}, q^{\sim}) \in D[X]$. $t_1 \wedge t, t_1 \vee t$ are any one of 1,0 (or 2), $\Gamma_x(p,q)$

$\therefore t_1 \wedge t, t_1 \vee t \in D[X]$

b. t_1 is a reduced tree on X.By the previous lemma $t_1^{\sim} \to t_1$ interchanges 1 and 0.

$X[t^{\sim}(x \leftarrow M)] = X(t^{\sim}) - x$ if t_1 is any one of 1 and 0 $X(t_1), X(t_1^{\sim})$ are variables of X $X(t_1^{\sim}) = X(t_1)$. If t_1 is 2 then $X(t_1^{\sim}) = X(t_1)$ .If t_1 is $\Gamma_x(p,q)$ then $X(t_1^{\sim}) = X(t_1)$ .If t_1 is any one of 1,0 (or 2) then $X(t_1 \wedge t) = X(t_1)$ If t_1 is $\Gamma_x(p,q)$ then

$$X[\Gamma_x(p,q) \wedge t] = X[\Gamma_x(p \wedge t(x \leftarrow M), q \wedge t(x \leftarrow M))]$$
$$= X[p \wedge t(x \leftarrow 1)]$$
$$= X[q \wedge t(x \leftarrow 0)]$$

Similarly for $X[t_1 \vee t]$. These are all subsets of $X(t_1) \cup X(t)$.

a. $\phi : D[X] \to D[X]$ is a homomorphism. If t is any one of 1,0 (or 2) then ϕ is ' homomorphism.

If $t_1 = \Gamma_x(p,q)$ then
$$\phi(t_1 \wedge t) = \phi[\Gamma_x\{p \wedge t(x \leftarrow M), q \wedge t(x \leftarrow M)\}]$$
$$= \Gamma_x\{\phi(p \wedge t)(x \leftarrow M), \phi(q \wedge t)(x \leftarrow M)\}$$
$$= \Gamma_x\{\phi(p) \wedge \phi(t), \phi(q) \wedge \phi(t)\}$$
$$= \Gamma_x\{\phi(p), \phi(q)\} \wedge \phi(t)$$
$$= \phi\{\Gamma_x \phi(p), \phi(q)\} \wedge \phi(t)$$
$$= \phi(t_1) \wedge \phi(t)$$

Similarly we can prove for $\phi(t_1 \vee t) = \phi(t_1) \vee \phi(t)$

$\Rightarrow \phi$ is homomorphism.

Definition2.6: Let X be a set. The function $\phi : D[X] \to A(A^X)$ is defined recursively as follows

a. $\phi(1), \phi(0)(and \phi(2))$ are the corresponding functions in $A(A^X)$
b. $\phi(\Gamma_x(p,q) = \Gamma_x(\phi(P), \phi(q))$

Lemma2.7: Let X be a set. Let $x \in X, M = \{0,1\}$ and $t \in D[X]$ then $\phi[t(x \leftarrow M)] = \phi(t)(x \leftarrow M)$

Proof: Induct on t .If t is one of 1,0 (or 2) then both sides evaluate to the corresponding constant functions in $A(A^X)$. If t is

$\Gamma_x(p,q)$
$$\phi(t)(x \leftarrow M) = \Gamma_x(\phi(p), \phi(q))(x \leftarrow 1)$$
$$= \phi(p)(x \leftarrow 1)$$
$$= \phi[p(x \leftarrow 1)]$$
$$= \phi(p)$$
$$= \phi(t(x \leftarrow 1))$$

Similarly for $\phi(t)(x \leftarrow 0)$

Proposition2.8: Let x be a set. Then ϕ is a ', $\wedge$ and $\vee$ homomorphism from D[X] onto F[X]`

Proof: We show that $\phi(t_1 \wedge t) = \phi(t_1) \wedge \phi(t)$ by induction on t. If t_1 1,0 (or 2) then it is an easy consequence of the properties of these elements and the corresponding constant functions in $A(A^X)$

If $t_1 = \Gamma_x(p,q)$ we have

$$\phi(t_1 \wedge t) = \phi[\Gamma_x\{p \wedge t(x \leftarrow 1), q \wedge t(x \leftarrow 0)\}]$$
$$= \Gamma_x\{\phi(p \wedge t)(x \leftarrow 1), \phi(q \wedge t)(x \leftarrow 0)\}$$
$$= \Gamma_x\{\phi(p) \wedge \phi(t), \phi(q) \wedge \phi(t)\}$$
$$= \Gamma_x\{\phi(p), \phi(q)\} \wedge \phi(t)$$
$$= \phi\{\Gamma_x \phi(p), \phi(q)\} \wedge \phi(t)$$
$$= \phi(t_1) \wedge \phi(t)$$

Similarly we can prove for $\phi(t_1 \vee t) = \phi(t_1) \vee \phi(t)$

$\Rightarrow \phi$ is homomorphism.

Finally ϕ maps D[X] onto F[X]`

$\because \phi(\Gamma_x(1,0)) = \Gamma_x(1,0) = x$. We remark that ϕ is not an isomorphism from D[X] onto F[X]. In order to understand the relationship between D[X]and F[X] closely we make the following

Definition2.9: . Let X be a set.The relation ~ on D[X] is defined to be the smallest equivalence relation on D[X] that satisfies the following

a. If $x, y \in X$ & $p,q,r,s \in D[X]$ satisfies $x \neq y$ & $x, y \notin X(p) \cup X(q) \cup X(r) \cup X(s)$ then $\Gamma_x[\Gamma_y(p,q), \Gamma_y(r,s)] \sim \Gamma_y[\Gamma_x(p,r), \Gamma_x(q,s)]$
b. $x \in X$ & $p,q,r,s \in D[X]$ satisfies $p \sim r, q \sim s$ & $x \notin X(p) \cup X(q) \cup X(r) \cup X(s)$ then $\Gamma_x(p,q) \sim \Gamma_x(r,s)$
c. When 2 is distinguished ;if $x \in X$ then $\Gamma_x(2,2) = 2$

Theorem 2.10: Let x be a set, $t_1, t_2 \in D[X]$ .If $t_1 \sim t_2$ then $\phi(t_1) = \phi(t_2)$

Proof: We know that $\Gamma_x[\Gamma_y(p,q), \Gamma_y(r,s)] \sim \Gamma_y[\Gamma_x(p,r), \Gamma_x(q,s)]$ and $\Gamma_x(2,2) = 2$

Let $t_1 = \Gamma_x(p,q)$ & $t_2 = \Gamma_y(r,s)$. If $t_1 \sim t_2$ then

$\Gamma_x(p,q) \sim \Gamma_y(r,s) \Rightarrow p \sim r, q \sim s$

$\phi(\Gamma_x(p,q)) \sim \phi(\Gamma_y(r,s))$

$\Rightarrow \Gamma_x(\phi(p), \phi(q)) \sim \Gamma_y(\phi(r), \phi(s))$

$\Gamma_x(\phi(p), \phi(q))(x \leftarrow 1) \sim \Gamma_y(\phi(r), \phi(s))(x \leftarrow 1)$

$\Rightarrow \phi(p) = \phi(r)$

$\Gamma_x(\phi(p), \phi(q))(x \leftarrow 0) \sim \Gamma_y(\phi(r), \phi(s))(x \leftarrow 0)$

$\Rightarrow \phi(q) = \phi(s)$

$\therefore \phi(t_1) = \phi(t_2)$

When U is distinguished $\phi : D[X] \to F[X]$ is homomorphism.

Theorem 2.11 : Let x be a set. Let $x \in X$ & $p,q,r,s \in D[X]$ *with* $x \notin X(p) \cup X(q)$

a. *If* $\phi(\Gamma_x(p,q))$ is a constant function then $\phi(p) = \phi(q) = U$
b. If $\phi(t) = 1$ *or* 0 then $t = 1$ *or* 0 respectively.
c. If $\phi(t) = U$ then $t \sim U$

Proof : a. We know that $\Gamma_2(p,q) = 2$ & $\Gamma_x(p,q)(f) = \Gamma_{f(x)}(p(f), q(f))$

If $f \in A^X$ satisfies $f(x)=2$ then $\phi(\Gamma_x(p,q)(f)=2$ this implies if $\phi(\Gamma_x(p,q))$ is a constant function then $\phi(\Gamma_x(p,q))=2$ If $\phi(p)\neq 2$ then there exist $f \in A^X$ such that $\phi(p)(f)\neq 2$ Define $g \in A^X$ by $g(y)=\begin{cases} f(y) & \text{if } y\in X-\{x\} \\ 1 & \text{if } y=x \end{cases}$

$$\Rightarrow \phi(\Gamma_x(p,q))(g)\neq 2 \Rightarrow \phi(\Gamma_x(p,q))\neq 2$$

$\Rightarrow \phi(\Gamma_x(p,q))$ is not a constant function. Similarly for $\phi(q)\neq 2$

$\therefore if \quad \phi(\Gamma_x(p,q))$ is a constant function then $\phi(p)=\phi(q)=2$

b. If $\phi(t)=1 \text{ or } 0$ then $t=1 \text{ or } 0$ respectively.

$$\phi(\Gamma_x(p,q))=\Gamma_x(\phi(p),\phi(q))$$
$$\Gamma_1(\phi(p),\phi(q))=\phi(p)(x\leftarrow 1)=1$$
$$\Rightarrow \phi(t)=1$$
$$\Gamma_0(\phi(p),\phi(q))=\phi(q)(x\leftarrow 0)=0$$
$$\Rightarrow \phi(t)=0$$

$\Rightarrow \phi(t)=1 \text{ or } 0 \text{ if } t=1 \text{ or } 0$ respectively.

c. $if \quad \phi(t)=2 \ \ then \ \ t\sim 2 \ \phi(\Gamma_x(p,q))=2 \text{ if } \quad \phi(p)=\phi(q)=2$

$$\phi(t)=2 \text{ if } t\sim 2$$

Undefined set of a reduced tree: Let X be a set. Let $t\in D[X]$

$$U(t)=\{x\in X \ / \text{ there exist } p,q\in D[X] \text{ with } t\sim\Gamma_x(p,q)\}$$

$U(t)$ is the undefined set of t because it satisfies following property

$$x\in U(t) \Leftrightarrow f\in A^X \quad if \ f(x)=2 \ \text{ then } \phi(t)(f)=2$$

References

[1] Koteswara Rao. P, A*-Algebra an If-Then-Else structures(thesis) 1994, Nagarjuna University, A.P., India
[2] Venkateswara Rao. J., On A*-Algebras(Thesis) 2000, Nagarjuna University, A.P., India
[3] Venkateswara Rao and SrinivasaRao.K, Pre A*-Algebra as a Poset, Africa Journal Mathematics and Computer Science Research.Vol.2 (4), , May 2009 pp 073-080.
[4] Venkateswara Rao, Tesfamariam, Habtu, Properties of Pre A*-function, Gazi University, Journal of science(2015). , 28 (2) :pp 239-244
[5] J.Venkateswara Rao, Tesfamariam, Habtu ,A Comprehensive study of Pre A*-function, Punjab University, Journal of Mathematics(ISSN 1016-2526), Vol. 46(1), 2014, pp. 67-75.
[6] Vijayabarathi.S. Srinivasa Rao. K, Pre A*-algebra function, Journal of Harmonized Research, 4(2),2016, pp 93-97.