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On The Hyperbola $y^2 = 3x^2 + 4$

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Abstract: The hyperbola represented by the binary quadratic equation $y^2 = 3x^2 + 4$ is analyzed for finding its non-zero distinct integer solutions. A few interesting relations among its solutions are presented. Also, knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Employing the solutions, a special Pythagorean triangle is constructed.

Keywords: Binary quadratic, Hyperbola, Parabola, Pell equation, Integral solutions.

I. Introduction

The binary quadratic Diophantine equations of the form $ax^2 - by^2 = N$, $(a, b, N \neq 0)$ are rich in variety and have been analyzed by many mathematicians for their respective integer solutions for particular values of a, b and N. In this context, one may refer [1-11].

This communication concerns with the problem of obtaining non-zero distinct integer solutions to the binary quadratic equation given by $y^2 = 3x^2 + 4$ representing hyperbola. A few interesting relations among its solutions are presented. Also, knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Employing the solutions, a special Pythagorean triangle is constructed.

NOTATIONS:

- Polygonal number of rank n with size m

$$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$$

- Pyramidal number of rank n with size m

$$P_n^m = \frac{1}{6}[n(n+1)][(m-2)n + (5-m)]$$

II. Method of Analysis

The binary quadratic equation representing hyperbola is given by

$$y^2 = 3x^2 + 4 \tag{1}$$

The smallest positive integer solution (x_0, y_0) of the above equation is

$$x_0 = 2, y_0 = 4$$

To obtain the other solutions of (1), consider the pellian equation

$$y^2 = 3x^2 + 1 \tag{2}$$

whose smallest positive integer solution is

$$\tilde{x}_0 = 1 \,,\; \tilde{y}_0 = 2 \tag{3}$$

The general solution $(\tilde{x}_n, \tilde{y}_n)$ of (2) is given by

$$\tilde{y}_n = \frac{1}{2} f_n \,,\; \tilde{x}_n = \frac{1}{2\sqrt{3}} g_n$$

where $f_n = (2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1}$, $g_n = (2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}$

Applying Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the general solution (x_{n+1}, y_{n+1}) of (1) is found to be

$$x_{n+1} = f_n + \frac{2}{\sqrt{3}} g_n \quad (4)$$

$$y_{n+1} = 2 f_n + \sqrt{3} g_n \quad (5)$$

Thus, (4) and (5) represent the integer solutions of the hyperbola (1). A few numerical examples are given in the following Table: 1

Table: 1 Numerical Examples

n	x_{n+1}	y_{n+1}
-1	2	4
0	8	14
1	30	52
2	112	194
3	418	724

Recurrence relations for x and y are given by

$$x_{n+3} - 4x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 4y_{n+2} + y_{n+1} = 0 \text{ , } n = -1, 0, 1, \ldots$$

A few interesting relations among the solutions are given below:

- $y_{n+1} = x_{n+2} - 2x_{n+1}$
- $y_{n+2} = 2x_{n+2} - x_{n+1}$
- $y_{n+3} = 7x_{n+2} - 2x_{n+1}$
- $y_{n+2} = 3x_{n+1} + 2y_{n+1}$
- $y_{n+3} = 2x_{n+3} - x_{n+2}$
- $x_{n+3} = 4y_{n+1} + 7x_{n+1}$
- $y_{n+2} = x_{n+3} - 2x_{n+2}$
- $y_{n+1} = 2x_{n+3} - 7x_{n+2}$

➢ Each of the following expressions represents a cubical integer:

- $2x_{3n+4} - 7x_{3n+3} + 3(2x_{n+2} - 7x_{n+1})$
- $2y_{3n+3} - 3x_{3n+3} + 3(2y_{n+1} - 3x_{n+1})$
- $7x_{3n+5} - 26x_{3n+4} + 3(7x_{n+3} - 26x_{n+2})$
- $7y_{3n+4} - 12x_{3n+4} + 3(7y_{n+2} - 12x_{n+2})$
- $13y_{3n+4} - 6x_{3n+5} + 3(13y_{n+2} - 6x_{n+3})$
- $\dfrac{3(15y_{n+1} - y_{n+3}) + 15y_{3n+3} - y_{3n+5}}{4}$

➢ Each of the following expressions represents the Nasty number:

- $6(2 + 2x_{2n+3} - 7x_{2n+2})$
- $6(2 + 2y_{2n+2} - 3x_{2n+2})$
- $6(2 + 7x_{2n+4} - 26x_{2n+3})$
- $6(2 + 7y_{2n+3} - 12x_{2n+3})$
- $6(2 + 13y_{2n+3} - 6x_{2n+4})$

- $\frac{3(8+15y_{2n+2}-y_{2n+4})}{2}$

- Each of the following expressions represents the bi-quadratic integer:
  - $6+4(2x_{2n+3}-7x_{2n+2})+2x_{4n+5}-7x_{4n+4}$
  - $6+4(2y_{2n+2}-3x_{2n+2})+2y_{2n+4}-3x_{4n+4}$
  - $6+4(7x_{2n+4}-26x_{2n+3})+7x_{4n+6}-26x_{4n+5}$
  - $6+4(7y_{2n+3}-12x_{2n+3})+7y_{4n+5}-12x_{4n+5}$
  - $6+4(13y_{2n+3}-6x_{2n+4})+13y_{4n+5}-6x_{4n+6}$
  - $6+4(15y_{2n+2}-y_{2n+4})+\frac{1}{4}(15y_{4n+4}-y_{4n+6})$

III. Remarkable Observations

1) Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in Table: 2 below:

Table: 2 Hyperbolas

S.NO	HYPERBOLAS	(X_n, Y_n)
1	$X_n^2-3Y_n^2=4$	$(2x_{n+2}-7x_{n+1}\ ,\ 4x_{n+1}-x_{n+2})$
2	$X_n^2-3Y_n^2=4$	$(2y_{n+1}-3x_{n+1}\ ,\ 2x_{n+1}-y_{n+1})$
3	$X_n^2-3Y_n^2=4$	$(7x_{n+3}-26x_{n+2}\ ,\ 15x_{n+2}-4x_{n+3})$
4	$X_n^2-3Y_n^2=4$	$(7y_{n+2}-12x_{n+2}\ ,7x_{n+2}-4y_{n+2})$
5	$4X_n^2-3Y_n^2=4$	$(13y_{n+2}-6x_{n+3}\ ,\ 7x_{n+3}-15y_{n+2})$
6	$3X_n^2-4Y_n^2=192$	$(15y_{n+1}-y_{n+3}\ ,\ y_{n+3}-13y_{n+1})$

2) Employing linear combination among the solutions for other choices of parabola which are presented in the Table: 3 below:

Table 3: Parabolas

S.NO	PARABOLAS	(X_n, Y_n)
1	$X_n-3Y_n^2=4$	$(2+2x_{2n+3}-7x_{2n+2}\ ,\ 4x_{n+1}-x_{n+2})$
2	$X_n-3Y_n^2=4$	$(2+2y_{2n+2}-3x_{2n+2}\ ,\ 2x_{n+1}-y_{n+1})$
3	$X_n-3Y_n^2=4$	$(2+7x_{2n+4}-26x_{2n+3}\ ,15x_{n+2}-4x_{n+3})$
4	$X_n-3Y_n^2=4$	$(2+7y_{2n+3}-12x_{2n+3}\ ,\ 7x_{n+2}-4y_{n+2})$
5	$4X_n-3Y_n^2=4$	$(2+13y_{2n+3}-6x_{2n+4}\ ,\ 7x_{n+3}-15y_{n+2})$
6	$3X_n-Y_n^2=48$	$(8+15y_{2n+2}-y_{2n+4}\ ,\ y_{n+3}-13y_{n+1})$

3) Let p,q be two non-zero distinct integers such that $p>q>0$, treat p,q as the generators of the Pythagorean triangle $T(\alpha,\beta,\gamma)$ where $\alpha=2pq, \beta=p^2-q^2, \gamma=p^2+q^2, p>q>0$ Taking $p=x_{n+1}+y_{n+1}, q=x_{n+1}$, it is observed that $T(\alpha,\beta,\gamma)$ is satisfied by the following relations.

- $3\beta - \gamma - 2\alpha = 8$
- $\frac{4A}{P} = \alpha + \beta - \gamma$
- $\frac{2A}{P} = x_{n+1} y_{n+1}$
- $3\left(\alpha - \frac{4A}{P}\right)$ is a nasty number.

4) **Relations between solutions and special polygonal numbers:**
   - $\left(P_y^5 * t_{3,x+1}\right)^2 = 27\left(P_x^3 * t_{3,y}\right)^2 + \left(2t_{3,y} * t_{3,x+1}\right)^2$
   - $\left(3P_y^3 * t_{3,x}\right)^2 = 3\left(P_x^5 * t_{3,y+1}\right)^2 + \left(2t_{3,y+1} * t_{3,x}\right)^2$
   - $36\left(P_{y-1}^4 * t_{3,x+1}\right)^2 = 27\left(P_x^3 * t_{3,2y-2}\right)^2 + \left(2t_{3,2y-2} * t_{3,x+1}\right)^2$
   - $\left(P_y^5 * t_{3,2x-2}\right)^2 = 108\left(P_{x-1}^4 * t_{3,y}\right)^2 + \left(2t_{3,y} * t_{3,2x-2}\right)^2$

IV. Conclusion

In this paper, we have presented infinitely many integer solutions for the Diophantine equation, represented by hyperbola given by $y^2 = 3x^2 + 4$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of equations and determine their integer solutions along with suitable properties.

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