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Observations on Two Special Hyperbolic Paraboloids

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Abstract: Knowing a solution of ternary quadratic diophantine equation representing hyperbolic paraboloid, a general formula for generating sequence of solutions based on the given solution is illustrated.
KEYWORDS: Ternary quadratic, generation of solutions, hyperbolic paraboloid

I. INTRODUCTION

The subject of diophantine equations in number theory has attracted many mathematicians since antiquity. It is well-known that a diophantine equation is a polynomial equation in two or more unknowns with integer coefficients for which integer solutions are required. An integer solution is a solution such that all the unknowns in the equation take integer values. An extension of ordinary integers into complex numbers is the gaussian integers. A gaussian integer is a complex number whose real and imaginary parts are both integers. It is quite obvious that diophantine equations are rich in variety and there are methods available to obtain solutions either in real integers or in gaussian integers.

A natural question that arises now is, whether a general formula for generating sequence of solutions based on the given solution can be obtained? In this context, one may refer [1-7]. The main thrust of this communication is to show that the answer to the above question is affirmative in the case of the following ternary quadratic diophantine equations, each representing a hyperbolic paraboloid.

II. METHOD OF ANALYSIS

Hyperbolic Paraboloid: 1

Consider the hyperbolic paraboloid given by

$$(a+1)x^2 - ay^2 = 2z \tag{1}$$

Introduction of the linear transformations

$$x = X \pm aT \ , \ y = X \pm (a+1)T \tag{2}$$

leads to

$$X^2 = (a^2 + a)T^2 + 2z$$

which is satisfied by

$$T = 4k \ , \ z = 2k^2 \Rightarrow X = 2k(2a+1)$$

In view of (2), we have

$$x = 8ka + 2k \ , 2k \ \ and \ \ y = 8ka + 6k \ , -2k \tag{3}$$

Denote the above values of x, y, z as x_0, y_0, z_0 respectively. We illustrate a process of obtaining sequence of integer solutions to the given equation based on its given solution (3).

Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = h - x_0 \ , \ y_1 = y_0 + h \ , \ z_1 = z_0 + h \tag{4}$$

in which h is an unknown to be determined.

Substitution of (4) in (1) gives

$$h = 2(a+1)x_0 + 2ay_0 + 2 \tag{5}$$

Using (5) in (4), the second solution (x_1, y_1, z_1) of (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M(x_0, y_0, z_0)^t$$

where t is the transpose and

$$M=\begin{pmatrix} 2a+1 & 2a & 0 & 2 \\ 2a+2 & 2a+1 & 0 & 2 \\ 2a+2 & 2a & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of (1) written in the matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \\ 1 \end{pmatrix} = \begin{pmatrix} Y_n & aX_n & 0 & X_n \\ (a+1)X_n & Y_n & 0 & \frac{Y_n-1}{a} \\ (a+1)X_n & Y_n-1 & 1 & \frac{Y_n-1}{a} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix} \quad , n=0,1,2,....$$

where (X_n, Y_n) is the general solution of the Pellian equation $Y^2=(a^2+a)X^2+1$

It is worth mentioning here that it is possible to obtain a different generation formula for the above equation as shown below:

Consider the transformation

$$x_1 = x_0 + h_0 \ , \ y_1 = y_0 + h_0 \ , \ z_1 = z_0 + h_0^2 \tag{6}$$

Substituting (6) in (1) and simplifying, we get

$$h_0 = 2(a+1)x_0 - 2ay_0$$

Substituting the value of h_0 in (6), we have

$$x_1 = (2a+3)x_0 - 2ay_0 \ , \ y_1 = (2a+2)x_0 - (2a-1)y_0 \ , \ z_1 = z_0 + ((2a+2)x_0 - 2ay_0)^2$$

Repeating the above process, the generation formula for (1) is represented by

$$x_n = (3^n(a+1) - a)x_0 + a(1-3^n)y_0$$
$$y_n = (3^n-1)(a+1)x_0 + (a+1-3^n a)y_0$$
$$z_n = z_0 + \frac{(9^n-1)}{8}[(2a+2)x_0 - 2ay_0]^2$$

Illustration:

Taking $a=2$, the generation formulae for the equation $3x^2-2y^2=2z$ are given by

$$x_{n+1} = Y_n x_0 + 2X_n y_0 + X_n \qquad\qquad x_n = (3(3^n)-2)x_0 + 2(1-3^n)y_0$$
$$y_{n+1} = 3X_n x_0 + Y_n y_0 + \frac{Y_n-1}{2} \qquad \text{and} \qquad y_n = 3(3^n-1)x_0 + (3-2(3^n))y_0$$
$$z_{n+1} = 3X_n x_0 + (Y_n-1)y_0 + z_0 + \frac{Y_n-1}{2} \qquad\qquad z_n = z_0 + \frac{(9^n-1)}{8}[6x_0-4y_0]^2$$

Hyperbolic Paraboloid: 2

Consider the hyperbolic paraboloid given by

$$(a+2)x^2 - ay^2 = 2z \tag{1}$$

Introduction of the linear transformations

$$x = X \pm aT \ , \ y = X \pm (a+2)T \tag{2}$$

leads to

$$X^2 = (a^2+2a)T^2 + z$$

which is satisfied by

$$T = k \ , \ z = k^2 \Rightarrow X = k(a+1)$$

In view of (2), we have

$$x = k(2a+1), \ k \ \ and \ \ y = (2a+3)k \ , -k \tag{3}$$

Denote the above values of x, y, z as x_0, y_0, z_0 respectively. We illustrate a process of obtaining sequence of integer solutions to the given equation based on its given solution (3).

Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = h - x_0 \ , \ y_1 = y_0 + h \ , \ z_1 = z_0 + h \qquad (4)$$

in which h is an unknown to be determined.
Substitution of (4) in (1) gives

$$h = (a+2)x_0 + ay_0 + 1 \qquad (5)$$

Using (5) in (4), the second solution (x_1, y_1, z_1) of (1) is expressed in the matrix form as

$$(x_1, y_1, z_1)^t = M(x_0, y_0, z_0)^t$$

where t is the transpose and

$$M = \begin{pmatrix} a+1 & a & 0 & 1 \\ a+2 & a+1 & 0 & 1 \\ a+2 & a & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of (1) written in the matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \\ 1 \end{pmatrix} = \begin{pmatrix} Y_n & aX_n & 0 & X_n \\ (a+2)X_n & Y_n & 0 & \frac{Y_n - 1}{a} \\ (a+2)X_n & Y_n - 1 & 1 & \frac{Y_n - 1}{a} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{pmatrix} \quad , n = 0, 1, 2, \ldots.$$

where (X_n, Y_n) is the general solution of the Pellian equation $Y^2 = (a^2 + 2a)X^2 + 1$

It is worth mentioning here that it is possible to obtain a different generation formula for the above equation as shown below:
Consider the transformation

$$x_1 = x_0 + h_0 \ , \ y_1 = y_0 + h_0 \ , \ z_1 = z_0 + 2h_0^2 \qquad (6)$$

Substituting (6) in (1) and simplifying, we get

$$h_0 = (a+2)x_0 - ay_0$$

Substituting the value of h_0 in (6), we have

$$x_1 = (a+3)x_0 - ay_0 \ , \ y_1 = (a+2)x_0 - (a-1)y_0 \ , \ z_1 = z_0 + 2((a+2)x_0 - ay_0)^2$$

Repeating the above process, the generation formula for (1) is represented by

$$x_n = \frac{1}{2}\left[(3^n(a+2) - a)x_0 + a(1 - 3^n)y_0\right]$$

$$y_n = \frac{1}{2}\left[(3^n - 1)(a+2)x_0 + (a + 2 - (3^n)a)y_0\right]$$

$$z_n = z_0 + \frac{(9^n - 1)}{4}\left[(a+2)x_0 - ay_0\right]^2$$

Illustration:

Taking $a = 3$, the generation formulae for the equation $5x^2 - 3y^2 = 2z$ are given by

$$x_{n+1} = Y_n x_0 + 3X_n y_0 + X_n$$

$$y_{n+1} = 5X_n x_0 + Y_n y_0 + \frac{Y_n - 1}{3}$$

$$z_{n+1} = 5X_n x_0 + (Y_n - 1)y_0 + z_0 + \frac{Y_n - 1}{3}$$

and

$$x_n = \frac{1}{2}\left[(5(3^n) - 3)x_0 + 3(1 - 3^n)y_0\right]$$

$$y_n = \frac{1}{2}\left[5(3^n - 1)x_0 + (5 - 3(3^n))y_0\right]$$

$$z_n = z_0 + \frac{(9^n - 1)}{4}\left[5x_0 - 3y_0\right]^2$$

Note: Observe that the general formula for the generation of solutions is not unique.

To conclude, one may attempt for obtaining generation formula for other choices of hyperbolic paraboloid

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