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Uniform Stability of a class of impulsive fractional *q*-difference systems with infinite delay

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Abstract: In this manuscript, using Lyapunov's direct method and Razumikhin techniques, the uniform stability of impulsive fractional q-difference systems with infinite delay is studied. The conditions for uniform stability are discussed.

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Keywords: impulsive fractional *q*-difference equations; uniform stability; Lyapunov's direct method

1. Introduction

The fractional calculus deals with the generalization of integration and differentiation of any order. Because of its distinguished applications in various branches of science and engineering, there has been a great deal of interest in this field [1–6]. An important and new issue is to combine the time scales [7] and fractional calculus [8–11] looking for a better description of the phenomena having both discrete and continuous behaviors.

Accompanied with the development of the theory on fractional q-calculus, The boundary value problem of fractional *q*-difference equations and impulsive fractional *q*-difference equations was studied in many reports[12–17]. In recent years, many results have been obtained in the stability theory of impulsive differential equations with infinite delays [18–19]. the stability of q-fractional dynamic systems has attracted the attention of several researchers[20]. But, the stability results for impulsive fractional q-difference systems with infinite delay are scarce. The present paper is inspired by[18–19], we extend the method of Lyapunov functions to study the uniform stability of solutions of the following impulsive fractional q-difference system with infinite delay:

$$\begin{cases} {}_q^C\nabla_{t_0}^{\alpha} x(t) = f(t, x_t), \quad t \ge t_0, t \ne \tau_k \\ x(\tau_k) = I_k(x(\tau_k^-)) + J_k(x(\mu\tau_k^-)), \quad k \in \mathrm{N}_1 \end{cases} \tag{1.1}$$

where $0 < q < 1$, $0 < \alpha < 1$, ${}_q^C\nabla_{t_0}^{\alpha}$ denotes the left Caputo q-fractional derivative of order α ,let T_q be the time scale [7] $T_q = \{q^n : n \in \mathrm{Z}\} \bigcup \{0\}$. $t, t_0, \mu \in T_q$, t_0 and μ are constants, $0<\mu<1$, $x \in \mathbf{R}^n$, $f \in C[T_q \times D, \mathrm{R}^n]$,

$I_k, J_k \in C(\mathrm{R}, \mathrm{R}), k = 1, 2, 3, \ldots, D$ is an open set in $PC([\mu,1]_{T_q}, \mathrm{R}^n)$, where $[\mu,1]_{T_q} = [\mu,1] \bigcap T_q$, $PC([\mu,1]_{T_q}, \mathrm{R}^n)$ denotes the set of piecewise right continuous functions $\phi : [\mu,1]_{T_q} \to \mathbf{R}^n$ with the sup-norm $|\phi| = \sup_{\lambda \in [\mu,1]_{T_q}} \|\phi(\lambda)\|$,

where $\|\cdot\|$ is a norm in $\mathbf{R}^n$. For each $t \ge t_0$ $x_t \in PC([\mu,1]_{T_q}, \mathrm{R}^n)$ is defined by $x_t(\lambda) = x(\lambda t)$, $\mu \le \lambda \le 1$.

Let $\tau_k \in T_q$, $k = 0, 1, 2, \ldots$, and $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots$, $\tau_k \to +\infty$ for $k \to +\infty$, $x(t^+) = \lim_{s \to t^+} x(s)$, and $x(t^-) = \lim_{s\to t^-} x(s)$. A function $x(t)$ is called a solution of (1.1) with the initial condition

$$x_\sigma = x(\lambda\sigma) = \varphi(\lambda), \quad \lambda \in [\mu, 1]_{T_q} \quad , \tag{1.2}$$

where $\sigma \in T_q, \sigma \ge t_0$ and $\varphi \in PC([\mu,1]_{T_q}, \mathrm{R}^n)$, if it satisfies both (1.1) and (1.2).

2. Preliminaries

In this section we summarize the basic definitions and properties of q-calculus and fractional q-integrals and derivatives. For more details on the theory of q-calculus we refer to [21] and for the theory of q-fractional calculus we refer to [10,11](and the references therein).

For $0 < q < 1$, let the time scale[7] $T_q = \{q^n : n \in Z\} \bigcup \{0\}$.

For a function $f : T_q \to \mathbf{R}$, the nabla q-derivative of f is given by

$$\nabla_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in T_q - \{0\}.$$

The nabla q-integral of f on the interval $[0, t]$ is given by

$$\int_0^t f(s)\nabla_q s = (1-q)t \sum_{n=0}^{\infty} f(tq^n) q^n$$

and on for $[a, t]$, $a \in T_q$ is given by

$$\int_a^t f(s)\nabla_q s = \int_0^t f(s)\nabla_q s - \int_0^a f(s)\nabla_q s\,.$$

Moreover

$$\int_t^\infty f(s)\nabla_q s = (1-q)t \sum_{n=1}^{\infty} f(tq^{-n}) q^{-n},$$

and for $0 < b < \infty$ in T_q

$$\int_t^b f(s)\nabla_q s = \int_t^\infty f(s)\nabla_q s - \int_b^\infty f(s)\nabla_q s\,.$$

The fundamental theorem in q-calculus gives

$$\nabla_q \int_0^t f(s)\nabla_q s = f(t)$$

and if f is continuous at 0,

$$\int_0^t \nabla_q f(s)\nabla_q s = f(t) - f(0)\,.$$

The q-factorial function is defined by $(t-s)_q^n = \prod_{k=0}^{n-1}(t - sq^k), n \in \mathrm{N}$, and for $\alpha \ne 1,2,3\ldots$, the q-factorial function has the following form

$$(t-s)_q^\alpha = t^\alpha \prod_{n=0}^{\infty} \frac{t - sq^n}{t - sq^{\alpha+n}}, \alpha \in \mathrm{R}\,.$$

The *q*-gamma function, $\Gamma_q(\alpha)$ for $\alpha \in \mathrm{R} \setminus \{0,-1,-2,\cdots\}$ is defined by $\Gamma_q(\alpha)=\frac{(1-q)_q^{\alpha-1}}{(1-q)^{\alpha-1}}$,

The *q*-gamma function satisfies the identity $\Gamma_q(\alpha+1)=\frac{1-q^a}{1-q}\Gamma_q(\alpha),\ \Gamma_q(1)=1,\ \alpha>0$.

The left *q*-fractional integral of order $\alpha>0$, ${}_qI_a^\alpha$ a starting from $0<a\in T_q$ is defined by

$${}_qI_a^\alpha f(t)=\frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1} f(s)\nabla_q s$$

When $\alpha=n\in\mathrm{N}$, we have $\nabla_{qq}^n I_a^n f(t)=f(t)$ for $0\le a\in T_q$ .It is worth mentioning that the left *q*-fractional integral ${}_qI_a^\alpha$ a maps functions defined T_q to functions defined on T_q.

The left Caputo *q*-fractional derivative of order $\alpha>0, \alpha\notin\mathrm{N}$ of a function *f* is defined by

$${}_q^C\nabla_a^\alpha f(t)={}_q I_a^{(n-\alpha)}\nabla_q^n f(t)=\frac{1}{\Gamma_q(n-\alpha)}\int_a^t (t-qs)_q^{n-\alpha-1}\nabla_q^n f(s)\nabla_q s,$$

where $n=[\alpha]+1$. Here $[\alpha]$ is the greatest integer less than α.

Property 2.1 ([11]). Assume $\alpha>0$ and *f* is defined in suitable domains. Then,

$${}_qI_a^\alpha {}_q^C\nabla_a^\alpha f(t)=f(t)-\sum_{k=0}^{n-1}\frac{(t-a)_q^k}{\Gamma_q(k+1)}\nabla_q^k f(a) \quad ,$$

and if $0<\alpha\le 1$ then ${}_qI_a^\alpha {}_q^C\nabla_a^\alpha f(t)=f(t)-f(a)$.

Throughout this paper we let the following hypotheses hold:

(H_1) For $t\in[\mu\sigma,\sigma]_{T_q}$, the solution $x(t,\sigma,\varphi)$ coincides with the function $\varphi(\frac{t}{\sigma})$.

(H_2) For each function $x(s):[\mu\sigma,\infty]_{T_q}\to\mathrm{R}^n$, $x(\tau_k^-),x(\tau_k^+)$ exist and, $x(\tau_k^+)=x(\tau_k)$, $f(t,x_t)$ is continuous for almost all $t\in[\sigma,+\infty]_{T_q}$ and at the discontinuous points *f* is right continuous.

(H_3) $f(t,\phi)$ is Lipschitzian in ϕ in each compact set in $PC([\mu,1]_{T_q},\mathrm{R}^n)$.

(H_4) The functions $I_k,J_k,k=1,2,\ldots$,are such that if $x\in D, I_k\ne 0$, and $J_k\ne 0$, then $I_k(x(t))+J_k(x(\mu t))\in D$.

(H_5) $f(t,0)\equiv 0, I_k(0)\equiv 0$ and $J_k(0)\equiv 0, k=1,2,\ldots$, so that $x(t)\equiv 0$ is a solution of (1.1), which we call the zero solution.

In this paper, we assume that $f(t,x_t)$, I_k and J_k satisfy certain conditions such that the solution of systems (1.1) and (1.2) exists on $[\sigma,\infty]_{T_q}$ and is unique. We using the following notation:

$S(\rho)=\{x\in \mathrm{R}^n : \|x\|<\rho\}$,

$PC(\rho)=\{\phi\in PC([\mu,1]_{T_q},\mathrm{R}^n):\phi=x_\sigma,|\phi|<\rho\}$;

$PCB(t)=\{x_t\in D: x_t \text{ is bounded}\}$;

$PCB_\rho(\sigma)=\{\phi\in PCB(\sigma):|\phi|<\rho\}$

Definition 2.2 The zero solution of the system (1) is said to be:

(D_1) stable, if for any $\sigma\geq t_0,\sigma\in T_q$ and $\varepsilon>0$, there exists a $\delta=\delta(\varepsilon,\sigma)>0$ such that $\phi\in PC(\delta)$ implies that

$\|x(t;\sigma,\varphi)\|<\varepsilon$, for all $t\in T_q, t\geq\sigma$.

(D_2) uniformly stable, if it is stable and δ depends only on ε.

Definition 2.3 The function $V:[t_0,+\infty)_{T_q}\times S(\rho)\to \mathrm{R}^+$ belongs to class v_0 if:

(1) the function V is continuous on each of the sets $[\tau_{k-1},\tau_k)_{T_q}\times S(\rho)$ and for all $t\geq t_0, V(t,0)\equiv 0$;

(2) $V(t,x)$ is locally Lipschitzian in $x\in S(\rho)$;

(3) for each $k=1,2,\ldots$, there exist finite limits

$\lim_{(t,y)\to(\tau_k^-,x)} V(t,y)=V(\tau_k^-,x)$, $\lim_{(t,y)\to(\tau_k^+,x)} V(t,y)=V(\tau_k^+,x)$,

with $V(\tau_k^+,x)=V(\tau_k,x)$ satisfied.

3. Main results

In this part, we consider the uniform stability of the impulsive fractional *q*-difference system with infinite delay(1.1). We have the following two theorems about the uniform stability of the system (1.1).

Let the sets *K* be defined as

$K=\{\omega\in C(\mathrm{R}^+,\mathrm{R}^+): \text{strictly increasing and } \omega(0)=0\}$;

$K_1=\{\omega\in C(\mathrm{R}^+,\mathrm{R}^+): \omega(0)=0 \text{ and } \omega(s)>0 \text{ for } s>0\}$.

Theorem 3.1 *Assume that there exist functions* $a,b\in K, V(t,x(t))\in v_0$ *such that*

(i) $a(\|x(t)\|)\leq V(t,x(t))\leq b(\|x(t)\|)$, *for all* $(t,x)\in[\mu t_0,+\infty)_{T_q}\times S(\rho)$;

(ii) $\nabla_q V(t,x(t))<0$;

(iii) $V(\tau_k,\ I_k(x(\tau_k^-))+J_k(x(\mu\tau_k^-)))\leq\frac{1+b_k}{2}[V(\tau_k^-,\ x(\tau_k^-))+V(\mu\tau_k^-,\ x(\mu\tau_k^-))]$, *where* $b_k\geq 0$, *and* $\sum_{k=1}^{\infty} b_k<\infty$.

Then the zero solution of (1.1) *is uniformly stable.*

Proof. Since $\sum_{k=1}^{\infty} b_k < \infty$, it follows that $\prod_{k=1}^{\infty}(1+b_k) = M$; obviously $1 \le M < \infty$. For any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\delta < b^{-1}(\frac{a(\varepsilon)}{M})$. We will prove that if $\varphi \in PC(\delta)$ then $\|x(t;\sigma,\varphi)\| < \varepsilon$ for $t \in [\sigma,+\infty]_{T_q}$.

Let $x(t) = x(t;\sigma,\varphi)$ denote the solution through (σ,φ). Let $\sigma \in [\tau_{m-1},\tau_m)_{T_q}$ for some $m \in \mathbf{N}$. Then, we will prove that

$$V(t,x(t)) \le b(\delta), \quad t \in [\sigma,\tau_m)_{T_q}. \qquad (3.1)$$

Obviously, for $t \in [\mu\sigma,\sigma]_{T_q}$, there exists an $\lambda \in [\mu,1]_{T_q}$ such that $t = \lambda\sigma$; then

$$V(t,x(t)) = V(\lambda\sigma, x(\lambda\sigma)) \le b(\|x(\lambda\sigma)\|) \le b(\|\varphi\|) \le b(\delta).$$

So if inequality (3.1) does not hold, then there exists an $\hat{r} \in [\sigma,\tau_m)_{T_q}$, such that

$$V(\hat{r},x(\hat{r})) > b(\delta),$$

$$V(t,x(t)) \le b(\delta), \quad t \in [\mu\sigma,\hat{r}]_{T_q}.$$

$$\nabla_q V(\hat{r},x(\hat{r})) \ge 0.$$

This contradicts condition (ii), so (3.1) holds. In view of inequality (3.1) and condition (iii), we have

$$V(\tau_m,x(\tau_m)) = V(\tau_m, I_m(x(\tau_m^-)) + J_m(x(\mu\tau_m^-))) \le \frac{1+b_m}{2}[V(\tau_m^-,x(\tau_m^-)) + V(\mu\tau_m^-,x(\mu\tau_m^-))] \le (1+b_m)b(\delta).$$

Next we prove that

$$V(t,x(t)) \le (1+b_m)b(\delta), \quad t \in [\tau_{m-1},\tau_m)_{T_q}. \qquad (3.2)$$

If this does not hold, then there exists an $\hat{s} \in [\tau_{m-1},\tau_m)_{T_q}$ such that

$$V(\hat{s},x(\hat{s})) > (1+b_m)b(\delta)$$

$$V(t,x(t)) \le (1+b_m)b(\delta), \quad t \in [\mu\sigma,\hat{s}]_{T_q}.$$

$$\nabla_q V(\hat{s},x(\hat{s})) \ge 0.$$

This contradicts condition (ii), so (3.2) holds. In view of inequality (3.2) and condition (iii), we have

$$\begin{aligned} V(\tau_{m+1},x(\tau_{m+1})) &= V(\tau_{m+1}, I_{m+1}(x(\tau_{m+1}^-)) + J_{m+1}(x(\mu\tau_{m+1}^-))) \\ &\le \frac{1+b_{m+1}}{2}[V(\tau_{m+1}^-,x(\tau_{m+1}^-)) + V(\mu\tau_{m+1}^-,x(\mu\tau_{m+1}^-))] \\ &\le (1+b_{m+1})(1+b_m)b(\delta). \end{aligned}$$

By simple induction, we can prove, in general, that for $k = 0,1,2,\ldots$

$$V(t,x(t)) \le (1+b_{m+k})\cdots(1+b_m)b(\delta), \quad t \in [\tau_{m+k},\tau_{m+k+1})_{T_q}.$$

$$V(\tau_{m+k+1}, x(\tau_{m+k+1})) \le (1+b_{m+k+1})(1+b_{m+k})\cdots(1+b_m)b(\delta).$$

This together with inequality (3.1) yields

$$V(t, x(t)) \le Mb(\delta), \quad t \in [\sigma, \infty)_{T_q}.$$

From this and condition (i) we have

$$a(\|x(t)\|) \le V(t, x(t)) \le Mb(\delta) < a(\varepsilon), \ t \in [\sigma, \infty)_{T_q}.$$

So $\|x(t)\| < \varepsilon, \ t \in [\sigma, \infty)_{T_q}$.

The zero solution of (1.1) is uniformly stable. The proof of Theorem 3.1 is completed.

Theorem 3.2 *Assume that there exist functions* $a, b, G \in K,\ P, H \in K_1,\ h, g \in PC(\mathrm{R}^+, \mathrm{R}^+),\ V(t, x(t)) \in v_0$ and H *is decreasing. For any* $\rho > 0$, *there exists a* $\rho_1 \in (0, \rho)$ *such that* $x \in S(\rho_1)$ *implies that* $I_k(x(t)) + J_k(x(\mu t)) \in S(\rho)$, *constants* $\beta_k \ge 0, k \in \mathrm{Z}^+$, *such that*

(i) $a(\|x(t)\|) \le V(t, x) \le b(\|x(t)\|)$, *for all* $(t, x) \in [\mu t_0, +\infty)_{T_q} \times \mathbf{R}^n$;

(ii) *For any* $(\tau_k, \psi) \in T_q \times PC([\mu, 1]_{T_q}, S(\rho_1))$, $V(\tau_k,\ I_k(x(\tau_k^-)) + J_k(x(\mu\tau_k^-))) \le (1+\beta_k)V(\tau_k^-,\ x(\tau_k^-))$, *where* $\sum_{k=1}^{\infty} \beta_k < \infty$;

(iii) *For any* $\sigma \in [t_0, +\infty)_{T_q}$ *and* $\psi \in PC([\mu, 1]_{T_q}, S(\rho))$, *if* $P(V(t, x(t))) > V(\lambda t, x(\lambda t))$ *for all* $\mu \le \lambda \le 1$, $t \in [\tau_{k-1}, \tau_k)_{T_q}, k \in \mathrm{Z}^+$ *then*

$$\nabla_q V(t, x(t)) \le h(t)H(V(t, x(t))) - g(t)G(V(t, x(t))),\ t \in [\tau_{k-1}, \tau_k)_{T_q}, k \in \mathrm{Z}^+,$$

where $\sup_{t \ge 0} h(t) < \infty$ *and* $P(s) > s$ *for* $s > 0$;

(iv) $\inf_{t \ge 0}\{g(t) - \gamma h(t)\} > 0$, *where* $\gamma = \lim_{s \to 0^+} \frac{H(s)}{G(s)} < \infty$.

Then the zero solution of (1.1) *is uniformly stable.*

Proof. Since $a \in K$, from condition (iii) and (iv), one may choose a small enough $\delta^* \in (0, \rho_1)$ such that

$$g(t) > h(t)\frac{H(s)}{G(s)} \quad \text{holds for all } t \ge 0 \text{ and } s \in (0, a(\delta^*)). \tag{3.3}$$

In fact, since $\gamma = \lim_{s \to 0^+} \frac{H(s)}{G(s)} < \infty$, we know that for any given $\varepsilon' > 0$, there exists a $\delta' = \delta'(\varepsilon') > 0$ such that $\gamma - \varepsilon' < \frac{H(s)}{G(s)} < \gamma + \varepsilon', s \in (0, \delta')$. In particular, let $\varepsilon' = \frac{\eta}{2M}$, where $\eta = \inf_{t \ge 0}\{g(t) - \gamma h(t)\} > 0$ and

$M=\sup_{t\geq 0} h(t)<\infty$. Then there exists a small enough $\delta'=\delta'(\eta,M)>0$ such that

$$\gamma-\frac{\eta}{2M}<\frac{H(s)}{G(s)}<\gamma+\frac{\eta}{2M}, s\in(0,\delta').$$

Note that $a\in K$, one may further choose a small enough $\delta^*\in(0,\rho_1)$ such that $a(\delta^*)<\delta'$.

Hence, it can be deduced that

$$g(t)\geq\gamma h(t)+\eta>h(t)(\gamma+\frac{\eta}{2M})>h(t)\frac{H(s)}{G(s)}$$

for all $t\geq 0$ and $s\in(0,a(\delta^*))$.

For any $\sigma\in[t_0,+\infty)_{T_q}$, let $x(t)=x(t;\sigma,\varphi)$ be a solution of (1.1) through (σ,φ). For any given $\varepsilon\in(0,\delta^*)$, one may choose a $\delta=\delta(\varepsilon)>0$ such that $b(\delta)<\beta^{-1}a(\varepsilon)$, where $\beta=\prod_{k=1}^{\infty}(1+\beta_k)$. Next we show that $\varphi\in PCB_\delta(\sigma)$

Implies $\|x(t)\|<\varepsilon$, $t\in[\sigma,+\infty)_{T_q}$. First, it is obvious that

$$a(\|x(t)\|)\leq V(t,x(t))\leq b(\|x(t)\|)\leq b(\delta)<\beta^{-1}a(\varepsilon),\ t\in[\mu\sigma,+\infty)_{T_q}. \quad (3.4)$$

Suppose that $\sigma\in[\tau_{m-1},\tau_m)_{T_q}$ for some $m\in Z^+$. Next we show that

$$V(t,x(t))\leq\beta^{-1}a(\varepsilon),\ t\in[\sigma,\tau_m)_{T_q}. \quad (3.5)$$

If this assertion is not true, then there exists some $t^*\in[\sigma,\tau_m)_{T_q}$ such that $V(t^*,x(t^*))>\beta^{-1}a(\varepsilon)$, and $V(t,x(t))\leq\beta^{-1}a(\varepsilon),\ t\in[\sigma,t^*]_{T_q}$, so $\nabla_q V(t^*,x(t^*))\geq 0$. Then it follows from (3.4) that

$$P(V(t^*,x(t^*)))>V(t^*,x(t^*))>\beta^{-1}a(\varepsilon)\geq V(s,x(s)), s\in[\mu t^*,t^*]_{T_q}. \quad (3.6)$$

Note that $\varepsilon<\delta^*<\rho_1$ and by (i), it can be deduced that

$$a(\|x(t)\|)\leq V(t,x(t))\leq\beta^{-1}a(\varepsilon)<a(\rho_1)<a(\rho),\ t\in[\mu t^*,t^*]_{T_q},$$

which implies that

$$\|x(t)\|<\rho_1<\rho,\ t\in[\mu t^*,t^*]_{T_q}. \quad (3.7)$$

By (3.3), (3.6), (3.7) and the fact that $\beta^{-1}a(\varepsilon)<a(\varepsilon)<a(\delta^*)$, using (iii) we obtain

$$\nabla_q V(t^*,x(t^*))\leq h(t^*)H(V(t^*,x(t^*)))-g(t^*)G(V(t^*,x(t^*)))$$

$$\leq h(t^*)H(\beta^{-1}a(\varepsilon)) - g(t^*)G(\beta^{-1}a(\varepsilon))$$

$$= G(\beta^{-1}a(\varepsilon))[h(t^*)\frac{H(\beta^{-1}a(\varepsilon))}{G(\beta^{-1}a(\varepsilon))} - g(t^*)] < 0 ,$$

which is a contradiction with $\nabla_q V(t^*, x(t^*)) \geq 0$ and thus (3.5) holds.

Considering (3.4) and (3.5), it can be deduce that $\|x(t)\| < \rho_1,\ t \in [\mu\tau_m, \tau_m)_{T_q}$, i.e., $x(t) \in PC([\mu\tau_m, \tau_m)_{T_q}, S(\rho_1)\}$.

Then by (ii), we have

$$V(\tau_m, x(\tau_m)) = V(\tau_m, I_m(x(\tau_m^-)) + J_m(x(\mu\tau_m^-)))$$
$$\leq (1+\beta_m)V(\tau_m^-, x(\tau_m^-)) \leq \beta^{-1}(1+\beta_m)a(\varepsilon).$$

By the same argument, we may prove that for $t \in [\tau_m, \tau_{m+1})_{T_q}$,

$$V(t, x(t)) \leq \beta^{-1}(1+\beta_m)a(\varepsilon).$$

By simple induction, we can prove that for $t \in [\sigma, \tau_m)_{T_q} \bigcup [\tau_k, \tau_{k+1})_{T_q}, k \geq m$,

$$V(t, x(t)) \leq \beta^{-1}(1+\beta_1)(1+\beta_2)\cdots(1+\beta_k)a(\varepsilon) \leq a(\varepsilon) ,$$

which implies that

$$a(\|x(t)\|) \leq V(t, x(t)) \leq a(\varepsilon),\ t \in [\sigma, \infty)_{T_q} .$$

So $\|x(t)\| \leq \varepsilon,\ t \in [\sigma, \infty)_{T_q}$. The proof of Theorem 3.2 is complete.

References:

[1]. R.L. Bagley, P. Torvik: On the appearance of the fractional derivative in the behavior of real materials, J. Appl. Mech. 51, 294–298 (1984).

[2]. G. Samko, A.A. Kilbas, O.I. Marichev: Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon, 1993.

[3]. I. Podlubny: Fractional Differential Equations, Academic Press, San Diego, 1999.

[4]. B.J. West, M. Bologna, P. Grogolini: Physics of Fractal Operators, Springer, New York, 2003.

[5]. A.A. Kilbas, H.M. Sirvastava, J.J. Trujillo: Theory and Applications of Fractional Differential Equations, Elsevier B.V., 2006.

[6]. R.L. Magin: Fractional Calculus in Bioengineering, Begell House Publisher, Inc., Connecticut, 2006.

[7]. M. Bohner, A. Peterson: Dynamic Equations on Time Scales, Birkhäuser, Boston, 2001.

[8]. R.P. Agrawal: Certain fractional *q*-integrals and *q*-derivatives, Proc. Cambridge. Philos. Soc. 66, 365–370 (1969).

[9]. K.S. Miller, B. Ross: Fractional difference calculus, in: Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and Their Applications, Nihon University, Koriyama, Japan

May 1989, pp. 139–152, Ellis Horwood Ser. Math. App., Horwood, Chichester, 1989.

[10]. F.M. Atıcı, P.W. Eloe: Fractional *q*-calculus on a time scale, J. Nonlinear Math. Phys. 14 (3) ,333–344 (2007).

[11]. T. Abdeljawad, D. Baleanu:Caputo *q*-fractional initial value problems and a *q*-analogue Mittag-Leffler function, Commun. Nonlinear Sci. Numer. Simul. 16 (12), 4682–4688 (2011).

[12]. Ahmad B, Nieto J J, Alsaedi A, Al-Hutami H: Existence of solutions for nonlinear fractional *q*-difference integral equations with two fractional orders and nonlocal four-point boundary conditions. Journal of the Franklin Institute, 351:2890-2909(2014).

[13]. Agarwal R P, Ahmad B, Alsaedi A, Al-Hutami H: Existence Theory for *q*-Antiperiodic Boundary Value Problems of Sequential *q*-Fractional Integro differential Equations. Abstract and Applied Analysis, 2014: 1-12(2014).

[14]. J. Tariboon , S.K. Ntouyas , P. Agarwal: New concepts of fractional quantum calculus and applications to impulsive fractional *q* -difference equations. Adv. Differ. Equ. 2015,18 (2015).

[15]. Sitthiwirattham Thanin. On nonlocal fractional q-integral boundary value problems of fractional *q*-difference and fractional *q*-integrodifference equations involving different numbers of order and *q*.Boundary Value Problems, 2016(12):1-19(2016).

[16]. Bashir Ahmad , Sotiris K. Ntouyas , Jessada Tariboon, Ahmed Alsaedi , Hamed H. Alsulami: Impulsive fractional *q* -integro-difference equations with separated boundary conditions. Applied Mathematics and Computation, 281,199–213(2016) .

[17]. Shuyuan Wan, Yuqi Tang, Qi G.The existence of solutions for a class of impulsive fractional *q*-difference equations, European Journal of Mathematics and Computer Science,4(1):26-34(2017).

[18]. Yu Zhang, Jitao Sun:Stability of impulsive functional differential equations, Nonlinear Analysis, 68, 3665–3678(2008)

[19]. Xiaodi Li :Further analysis on uniform stability of impulsive infinite delay differential equations, Applied Mathematics Letters, 25,133–137 (2012).

[20]. Fahd Jarad, Thabet Abdeljawad, Dumitru Baleanu :Stability of *q*-fractional non-autonomous systems, Nonlinear Analysis: Real World Applications 14, 780–784 (2013).

[21]. T. Ernst, The history of *q*-calculus and new method (Licentiate Thesis), U.U.D.M. Report 2000: http://math.uu.se/thomas/Lics.pdf.