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# Negative solution to integral boundary value problem of p-Laplacian with fractional derivative

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**Abstract:** In this article, we consider the following integral boundary value problem of fractional differential equation with p-Laplacian operator:

$$\begin{aligned} &D^{\beta}(\varphi_p(D^{\alpha}u(t)))+f(t,u(t))=0, \quad 0<t<1,\\ &u(0)=\int_0^1 u(t)A(t)dt,\\ &D^{\alpha}u(0)=u(1)=0, \end{aligned}$$

where $1<\alpha\le 2$ , $0<\beta<1$ is a real number, $D^{\alpha}$ , $D^{\beta}$ is the conformable fractional derivative, $\varphi_p(s)=|s|^{p-2}s$ , $p>1, \varphi_p^{-1}=\varphi_q, \frac{1}{p}+\frac{1}{q}=1$, and $f$ : $[0,1]\times(-\infty,0]\to[0,+\infty)$ is continuous. $A(t)$ is a continuous function on $[0,1]$.

By the use of an approximation method and fixed point theorems on cone, some existence and multiplicity results of negative solutions are acquired.

**Keywords:** conformable fractional derivative; singular Green's function; fixed point theorems on cone; p-Laplacian operator

## 1. Introduction

Recently, differential equations have been of great interest. Integer order differential equations with p-Laplacian have been subject to a lot of research [1,2]. Now, many people pay attention to the existence and multiplicity of solutions for boundary value problems of fractional differential equations with p-Laplacian by the use of techniques of nonlinear analysis [3-6], upper and lower solutions method [7,8], coincidence degree [9], Banach contraction mapping principle [10], etc.

Chen *et al*. [9] investigated the boundary value problem for a fractional differential equation with a p-Laplacian operator at resonance,

$$\begin{aligned} &D^{\beta}(\varphi_p(D^{\alpha}u(t)))=f(t,u(t),D_{0+}^{\alpha}u(t)), \quad 0<t<1,\\ &D^{\alpha}u(0)=D^{\alpha}u(1)=0, \end{aligned}$$

where $0<\alpha,\beta\le 1, 1<\alpha+\beta\le 2$ , and $D_{0+}^{\alpha}$ , $D_{0+}^{\beta}$ is the Caputo fractional derivative.

$\varphi_p(s)=|s|^{p-2}s$ , $p>1, \varphi_p^{-1}=\varphi_q, \frac{1}{p}+\frac{1}{q}=1$, By using the coincidence degree theory, a new result of the existence of solution is obtained.

Lu *et al.* [7] studied the following p-Laplacian fractional differential equations boundary problems:

$$D^{\beta}(\varphi_p(D^{\alpha}u(t))) = f(t,u(t)), \quad 0<t<1,$$
$$u(0)=u'(0)=u'(1)=0, \quad D_{0+}^{\alpha}u(0)=D_{0+}^{\alpha}u(1)=0,$$

where $2<\alpha\le 3$ , $1<\beta\le 2$ , $D_{0+}^{\alpha}$ , $D_{0+}^{\beta}$ are the standard Riemann-Liouville fractional derivatives, $\varphi_p(s)=|s|^{p-2}s$ , $p>1, \varphi_p^{-1}=\varphi_q, \frac{1}{p}+\frac{1}{q}=1$, and $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous. By the properties of the Green's function, the Guo-Krasnosel'skii fixed point theorem, the Leggett-Williams fixed point theorem, and the upper and lower solutions method, some new existence results are obtained.

Dong *et al.* [18] studied the following p-Laplacian fractional differential equations boundary problems:

$$D^{\alpha}(\varphi_p(D^{\alpha}u(t))) = f(t,u(t)), \quad 0<t<1$$
$$u(0)=u(1)=D_{0+}^{\alpha}u(0)=D_{0+}^{\alpha}u(1)=0$$

where $1<\alpha\le 2$ , $0<\beta<1$ is a real number, $D^{\alpha}$ , $D^{\beta}$ is the conformable fractional derivative, $\varphi_p(s)=|s|^{p-2}s$ , $p>1, \varphi_p^{-1}=\varphi_q, \frac{1}{p}+\frac{1}{q}=1$, and $f$: $[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous. By the use of an approximation method and fixed point theorems on cone, some existence and multiplicity results of positive solutions are acquired.

To the best of our knowledge, there is a little literature about negative solution to integral boundary value problem of p-Laplacian with fractional derivative. Therefore, in order to fill this gap in the literature, in this paper, we investigate the following p-Laplacian fractional differential equation boundary value problem:

$$D^{\beta}(\varphi_p(D^{\alpha}u(t)))) + f(t,u(t)) = 0, \quad 0<t<1, \tag{1.1}$$
$$u(0)=\int_0^1 u(t)A(t)dt, \tag{1.2}$$
$$D^{\alpha}u(0)=u(1)=0, \tag{1.3}$$

where $1<\alpha\le 2$ , $0<\beta<1$ is a real number, $D^{\alpha}$ , $D^{\beta}$ is the conformable fractional derivative, $\varphi_p(s)=|s|^{p-2}s$ , $p>1$ , $\varphi_p^{-1}=\varphi_q$ , $\frac{1}{p}+\frac{1}{q}=1$ , and $f$ : $[0,1]\times[-\infty,0)\to[0,+\infty)$ is continuous. $A(t)$ is a continuous function on $[0,1]$. By the approximation method and fixed point theorems on cone, some existence and multiplicity results of positive solutions are obtained.

The rest of this paper is organized as follows. In Section 2, we recall some concepts relative to the new conformable fractional calculus and give some lemmas with respect to the corresponding Green's function. In Section 3, we investigate the existence of positive solution for the boundary value problem (1.1),(1.2),(1.3). In Section 4, the multiplicity of positive solutions is studied.

## 2. Preliminaries and lemmas

For the convenience of the reader, we give some background material from fractional calculus theory to facilitate the analysis of Problem(1.1),(1.2),(1.3).These results can be found in the recent literature; see [11-13].

**Definition 2.1** Let $\alpha \in (n, n+1]$ and $f$ be a *n*-differentiable function at $t > 0$, then the fractional conformable derivative of order $\alpha$ at $t > 0$ is given by

$$D^{\alpha} f(t) = D^{\alpha-n} f^{(n)}(t) = \lim_{\varepsilon \to 0} \frac{f^{(n)}(t + \varepsilon t^{n+1-\alpha}) - f^{(n)}(t)}{\varepsilon}, \tag{2.1}$$

provided the limit of the right hand side exists. If $f$ is $\alpha$ -differentiable in some $(0, a)$, $a > 0$,

and $\lim_{t \to 0+} D^{\alpha} f(t)$ exists, then define

$$D^{\alpha} f(0) = \lim_{t \to 0+} D^{\alpha} f(t). \tag{2.2}$$

**Remark 2.1** As a basic example, given $\alpha \in (n, n+1]$, we have

$$D^{\alpha} f(t) = 0,$$

where $k$ =0,1,...,$n$

**Lemma 2.1** ([18])Let $t > 0$, $\alpha \in (n, n+1]$. The function $f(t)$ is ($n$ +1)-differentiable if and only if $f$ is α-differentiable, moreover, $D^{\alpha} f(t) = t^{n+1-\alpha} f^{(n+1)}(t)$.

**Definition 2.2** ([11]) Let $\alpha \in (n, n+1]$ .The fractional integral of order $\alpha > 0$ at $t > 0$ of a function $f : (0, \infty) \to R$ is given by

$$I^{\alpha} f(t) = I^{n+1}(t^{\alpha-n-1} f(t)) = \frac{1}{n!} \int_0^t (t-s)^n s^{\alpha-n-1} f(s) ds \tag{2.3}$$

where $I^{n+1}$ denotes the integration operator of order $n$ +1.

**Lemma 2.2** ([18]) Let $\alpha \in (n, n+1]$ and $f$ be a continuous function defined in $(0, +\infty)$, one has

$D^{\alpha} I^{\alpha} f(t) = f(t)$ for $t > 0$.

**Lemma 2.3** ([13] Mean value theorem) Let $a > 0$ and $f : [a, b] \to R$ be a function with the properties that

(1) $f$ is continuous on $[a, b]$,

(2) $f$ is $\alpha$ -differentiable on $(a, b)$ for some $\alpha \in (0,1)$.

Then there exists $c \in (a,b)$ such that

$$D^{\alpha} f(c) = \frac{f(b) - f(a)}{\frac{1}{\alpha} b^{\alpha} - \frac{1}{\alpha} a^{\alpha}}.$$

**Lemma 2.4 (**[18]) Let $\alpha \in (n, n+1]$, $f$ be a $\alpha$-differentiable function at $t > 0$ ,then $D^{\alpha} f(t) = 0$ , for $t \in (0,+\infty)$ if and only if $f(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} + a_n t^n$ ,where $a_k \in R$ ,for $k = 0,1,\ldots,n.$

**Lemma 2.5 (**[18]) Assume that $u \in C(0,+\infty)$ with a fractional derivative of order $\alpha \in (n, n+1]$ that belongs to $C(0,1) \cap L(0,1)$ . Then

$$I^{\alpha} D^{\alpha} u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1} + c_n t^n \tag{2.4}$$

for some $c_k \in R$, $k = 0,1,\ldots,n.$

Now, we present the Green's function. In the following arguments, we always suppose that $\alpha \in (1,2]$.

**Lemma 2.6** Let $y \in C[0,1]$ and $\alpha \in (1,2]$ , $\beta \in (0,1)$ . $A(t)$ : $[0,1] \to [0,+\infty)$ is continuous and satisfy $\int_0^1 (1-t) A(t) dt < 1$

Then the fractional differential equation boundary value problem

$$D^{\beta} (\varphi_p (D^{\alpha} u(t))) + y(t) = 0, \quad 0 < t < 1, \tag{2.5}$$

$$u(0) = \int_0^1 u(t) A(t) dt, \tag{2.6}$$

$$D^{\alpha} u(0) = u(1) = 0, \tag{2.7}$$

has a unique solution,

$$u(t) = -\int_0^1 G(t,s) V(s) ds,$$

where

$$V(s) = \varphi_q (\int_0^t s^{\beta-1} y(s) ds)$$

$$G(t,s) = G_0(t,s) + \frac{1-t}{1-\Gamma} g_A(s)$$

with

$$G_0(t,s) = \begin{cases} s^{\alpha-1}(1-t), & 0 \le s \le t \le 1; \\ t(1-s)s^{\alpha-2}, & 0 \le t \le s \le 1. \end{cases}$$

$$\Gamma = \int_0^1 (1-t)A(t)dt, \qquad g_A(s) = \int_0^1 G_0(t,s)A(t)dt.$$

*Proof* Apply the operator $I^{\beta}$ on both sides of (2.5),with Lemma 2.5,we have

$$\varphi_p(D^{\alpha}u(t)) + C_0 = I^{\beta}(y(t)),$$

So ,

$$\begin{aligned}\varphi_p(D^{\alpha}u(t)) &= I^{\beta}(y(t)) - C_0 \\ &= \frac{1}{0!}\int_0^t (t-s)^0 s^{\beta-0-1} y(s)ds - C_0 \\ &= \int_0^t s^{\beta-1} y(s)ds - C_0.\end{aligned}$$

For some $C_0 \in R$ ,By the boundary conditions $D^{\alpha}u(0) = 0$ ,we have $C_0 = 0$ .Therefore,the unique solution of problem (2.5),(2.6),(2.7) is

$$\varphi_p(D^{\alpha}u(t)) = \int_0^t s^{\beta-1} y(s)ds.$$

Thus,the fractional differential equation boundary value problem (2.5),(2.6),(2.7) is equivalent to the problem

$$D^{\alpha}u(t) = \varphi_q(\int_0^t s^{\beta-1} y(s)ds), \tag{2.8}$$

$$u(1) = 0, \tag{2.9}$$

$$u(0) = \int_0^1 u(t)A(t)dt, \tag{2.10}$$

Apply the operator $I^{\alpha}$ on both sides of (2.8),with Lemma 2.5,we have

$$I^{\alpha}D^{\alpha}u(t) = u(t) + b_0 + b_1 t.$$

Then,

$$u(t) = I^{\alpha}\left(\varphi_q(\int_0^t s^{\beta-1} y(s)ds)\right) + b_0 + b_1 t = I^{\alpha}V(s) + b_0 + b_1 t$$

By the boundary conditions $u(1) = 0$ ,we have

$$u(1) = \int_0^1 (1-s)s^{\alpha-2}V(s)ds + b_0 + b_1 = 0,$$

$$u(0) = \int_0^0 (1-s)s^{\alpha-2}V(s)ds + b_0 = b_0 = \int_0^1 u(t)A(t)dt,$$

So

$$b_1 = -\int_0^1 (1-s)s^{\alpha-2}V(s) - b_0 = -\int_0^1 (1-s)s^{\alpha-2}V(s)ds - \int_0^1 u(t)A(t)dt.$$

Therefore,the unique solution of problem (2.8),(2.9),(2.10) is

$$u(t) = \int_0^t (t-s)s^{\alpha-2}V(s)ds + \int_0^1 u(t)A(t)dt - t\int_0^1 (1-s)s^{\alpha-2}V(s)ds - t\int_0^1 u(t)A(t)dt$$

$$= \int_0^t (t-s)s^{\alpha-2}V(s)ds - t\int_0^t (1-s)s^{\alpha-2}V(s)ds - t\int_t^1 (1-s)s^{\alpha-2}V(s)ds$$

$$+ (1-t)\int_0^1 u(t)A(t)dt$$

$$= \int_0^t [(t-s)-t(1-s)]s^{\alpha-2}V(s)ds - t\int_t^1 (1-s)s^{\alpha-2}V(s)ds \qquad + (1-t)\int_0^1 u(t)A(t)dt$$

$$= -(\int_0^t s(1-t)s^{\alpha-2}V(s)ds + t\int_t^1 (1-s)s^{\alpha-2}V(s)ds) + (1-t)\int_0^1 u(t)A(t)dt$$

Let

$$G_0(t,s) = \begin{cases} s^{\alpha-1}(1-t), & 0 \le s \le t \le 1; \\ t(1-s)s^{\alpha-2}, & 0 \le t \le s \le 1, \end{cases} \tag{2.11}$$

Then

$$u(t) = -\int_0^1 G_0(t,s)ds + (1-t)\int_0^1 u(t)A(t)dt. \tag{2.12}$$

Multiplying both sides of (2.12) by $A(t)$ and then integrating from 0 to 1,we have

$$\int_0^1 A(t)u(t)dt = -\int_0^1 A(t)\int_0^1 G_0(t,s)dsdt + \int_0^1 (1-t)A(t)\int_0^1 u(s)A(s)dsdt.$$

So

$$(1-\int_0^1 (1-t)A(t)dt)\int_0^1 u(t)A(t)dt = -\int_0^1 \int_0^1 A(t)G_0(t,s)dtV(s)ds. \tag{2.13}$$

Let

$$\Gamma = \int_0^1 (1-t)A(t)dt, \qquad g_A(s) = \int_0^1 G_0(t,s)A(t)dt. \tag{2.14}$$

From (2.13) and (2.14) we deduce that

$$\int_0^1 A(t)u(t)dt = -\frac{1}{1-\Gamma}\int_0^1 g_A(s)V(s)ds. \tag{2.15}$$

Substituting (2.15)into (2.12),we obtain

$$u(t) = -(\int_0^1 G_0(t,s)ds + \frac{1-t}{1-\Gamma}\int_0^1 g_A(s)V(s)ds) = -\int_0^1 G(t,s)V(s)ds.$$

Where $G(t,s) = G_0(t,s) + \frac{1-t}{1-\Gamma}g_A(s).$ (2.16)

The proof is complete. □

**Lemma 2.7** ([18])The function $G_0(t,s)$ defined by (2.11) satisfies the following properties:

(i) $G_0(t,s) > 0$, for all $t,s \in (0,1)$;

(ii) $\min_{\frac{1}{4}\le t\le\frac{3}{4}} G_0(t,s) \ge \frac{1}{4}\max_{0\le t\le 1} G_0(t,s) = \frac{1}{4}G_0(s,s)$, for $s \in (0,1)$.

It should be noted that the constant bound is new for fractional derivatives. It was pointed out that the Riemann-Liouville fractional derivative does not allow one to get a positive constant boundary (see [3], Remark2.2).

**Lemma 2.8** Let $0 \le \Gamma < 1$ and $g_A(s) \ge 0$ for $s \in [0,1]$. Then the Green function $G(t,s)$ defined by (2.2) satisfies

(1) $G:[0,1]\times[0,1] \to [0,\infty)$ is continuous.

(2) For any $t, s \in [0,1]$, we have

$$\frac{1}{4}\left(G_0(s,s) + \frac{1-t}{1-\Gamma} g_A(s)\right) \le G(t,s) \le G_0(s,s) + \frac{1-t}{1-\Gamma} g_A(s) ,$$

where $G_0(s,s) = (1-s)s^{\alpha-1}$.

*Proof* From (2.16) we have

$$\begin{aligned} G(t,s) &= G_0(t,s) + \frac{1-t}{1-\Gamma} g_A(s) \\ &\le G_0(s,s) + \frac{1-t}{1-\Gamma} g_A(s) \\ G(t,s) &= G_0(t,s) + \frac{1-t}{1-\Gamma} g_A(s) \\ &\ge \frac{1}{4} G_0(s,s) + \frac{1-t}{1-\Gamma} g_A(s) \\ &\ge \frac{1}{4}\left(G_0(s,s) + \frac{1-t}{1-\Gamma} g_A(s)\right) \end{aligned}$$

**Lemma 2.9** ([14])

(1) If $1 \le q \le 2$ ,then

$$\left|\varphi_q(u+v) - \varphi_q(u)\right| \le 2^{2-q}|v|^{q-1}$$

for all $u, v \in R$

(2) If $q > 2$ ,then

$$\left|\varphi_q(u+v) - \varphi_q(u)\right| \le (q-1)(|u|+|v|)^{q-2}|v|$$

for all $u, v \in R$

**Lemma 2.10** ([15]) Suppose $E$ is a Banach space and $T_n : E \to E$ , $n = 3,4,...$ are completely continuous operators, $T : E \to E$ . If $\|T_n u - Tu\|$ uniformly converges to zero when $n \to \infty$ for all bounded set $\Omega \subseteq E$ , then $T : E \to E$ is completely continuous.

**Definition 2.3** The map $\theta$ is said to be a nonnegative continuous concave functional on a cone $P$ of a Banach space $E$ provided that $\theta : P \to [0,\infty)$ is continuous and

$$\theta(tx+(1-t)y) \geq t\theta(x)+(1-t)\theta(y)$$

for all $x, y \in P$ and $0 < t < 1$.

The following fixed point theorems are useful in our proofs.

**Lemma 2.11** ([16]) Let $E$ be a Banach space, $P \subseteq E$ a cone, and $\Omega_1, \Omega_2$ two bounded open balls of $E$ centered at the origin with $\overline{\Omega}_1 \subseteq \Omega_2$. Suppose that $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either

(i) $\|Ax\| \leq \|x\|, x \in P \cap \partial\Omega_1$, and $\|Ax\| \geq \|x\|, x \in P \cap \partial\Omega_2$ or

(ii) $\|Ax\| \geq \|x\|, x \in P \cap \partial\Omega_1$, and $\|Ax\| \leq \|x\|, x \in P \cap \partial\Omega_2$

holds. Then A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

**Lemma 2.12** ([17]) Let $P$ be a cone in a real Banach space $E$, $P_C = \{x \in P \mid \|x\| \leq c\}$, $\theta$ a nonnegative continuous concave functional on $P$ such that $\theta(x) \leq \|x\|$, for all $x \in \overline{P_c}$, and

$P(\theta,b,d) = \{x \in P \mid b \leq \theta(x), \|x\| \leq d\}$. Suppose $A : \overline{P_c} \to \overline{P_c}$ is completely continuous and there exist constants $0 < a < b < d \leq c$ such that

(C1) $\{x \in P(\theta,b,d) \mid \theta(x) > b\}$ is non-empty, and $\theta(Ax) > b$, for $x \in P(\theta,b,d)$;

(C2) $\|Ax\| \leq a$, for $x \leq a$;

(C3) $\theta(Ax) > b$, for $x \in P(\theta,b,c)$ with $\|Ax\| > d$

Then A has at least three fixed points $x_1, x_2, x_3$ with

$$\|x_1\| \leq a,\ \ b < \theta(x_2),\ \ a < \|x_3\|,\ \ \theta(x_3) < b.$$

**Remark 2.2** ([17]) If we have $d = c$, then condition (C1) of Lemma 2.12 implies condition (C3) of Lemma 2.12.

## 3. Existence results

Let $E = C[0,1]$ be endowed with the ordering $\omega \leq v$ if $\omega(t) \leq v(t)$ for all $t \in [0,1]$, and the maximum norm $\|\omega\| = \max_{0\le t\le 1}|\omega(t)|$. Define

$$P = \{\omega \in E \,|\, \omega(t) \ge 0, \min_{\frac{1}{4}\le t\le \frac{3}{4}} \omega(t) \ge \frac{1}{4}\|\omega\|\}$$

Define the nonnegative continuous concave functional $\theta$ by

$$\theta(\omega) = \min_{\frac{1}{4}\le t\le \frac{3}{4}} \omega(t).$$

Given the continuous function $f \in C([0,1]\times(-\infty,0])$, define $T, T_n : P \to E$ as

$$(T\omega)(t) := \int_0^1 G(t,s)\varphi_q(\int_0^1 \tau^{\beta-1} f(\tau,-\omega(\tau)d\tau)ds$$

$$(T_n\omega)(t) := \int_{\frac{1}{n}}^1 G(t,s)\varphi_q(\int_{\frac{1}{n}}^1 \tau^{\beta-1} f(\tau,-\omega(\tau)d\tau)ds \text{ , } n = 3,4,....$$

If the operator $T$ has a fixed point $\omega$ ,According to Lemma 2.6 we have $u = -\omega$ is the solution of (1.1)-(1.3).

**Lemma 3.1** $T : P \to P$ is completely continuous.

*Proof* Firstly, we show that $T_n : P \to P$ are completely continuous for $n = 3,4,....$

Given $\omega \in P$, with Lemma2.8 and the nonnegativity of $f(t,-\omega)$, one has

$$(T_n\omega)(t) = \int_{\frac{1}{n}}^1 G(t,s)\varphi_q(\int_{\frac{1}{n}}^1 \tau^{\beta-1} f(\tau,-\omega(\tau)d\tau)ds$$

$$\le \int_{\frac{1}{n}}^1 \left( G_0(s,s) + \frac{1-t}{1-\Gamma} g_A(s) \right) \varphi_q(\int_{\frac{1}{n}}^1 \tau^{\beta-1} f(\tau,-\omega(\tau)d\tau)ds$$

so

$$\|T_n\omega\| \le \int_{\frac{1}{n}}^1 \left( G_0(s,s) + \frac{1-t}{1-\Gamma} g_A(s) \right) \varphi_q(\int_{\frac{1}{n}}^1 \tau^{\beta-1} f(\tau,-\omega(\tau)d\tau)ds$$

And next,if $\omega \in P$,

$$\min_{\frac{1}{4}\le t\le \frac{3}{4}}(T_n\omega)(t) = \min_{\frac{1}{4}\le t\le \frac{3}{4}} \int_{\frac{1}{n}}^1 G(t,s)\varphi_q(\int_{\frac{1}{n}}^1 \tau^{\beta-1} f(\tau,-\omega(\tau)d\tau)ds$$

$$\ge \frac{1}{4}\int_{\frac{1}{n}}^1 \left( G_0(s,s) + \frac{1-t}{1-\Gamma} g_A(s) \right) \varphi_q(\int_{\frac{1}{n}}^1 \tau^{\beta-1} f(\tau,-\omega(\tau)d\tau)ds$$

$$\ge \frac{1}{4}\|T_n\omega\|$$

As a consequence $T_n : P \to P$. The continuity of $T_n$ follows by the continuity of $G(t,s)$ and $f(t,-\omega)$.

Let $\Omega \in P$ be bounded, i.e., there exists a positive constant $M > 0$ such that $\|\omega\| < M$ for all $\omega \in \Omega$. Let

$$L = \max_{0 \le t \le 1, 0 \le \omega \le M} |f(t,-\omega(t))| + 1 \qquad H = \int_0^1 G_0(s,s)ds + 1$$

then,for $u \in \Omega$.we have

$$|(T_n\omega)(t)| = \left| \int_{\frac{1}{n}}^1 G(t,s)\varphi_q(\int_{\frac{1}{n}}^1 \tau^{\beta-1} f(\tau,-\omega(\tau)d\tau)ds \right|$$

$$\le \left(\frac{L}{\beta}\right)^{q-1} \int_{\frac{1}{n}}^1 \left( s^{\alpha-1}(1-s) + \frac{1}{1-\Gamma}\int_0^1 G_0(s,s)A(t)dt \right)ds$$

$$\le \left(\frac{L}{\beta}\right)^{q-1} \left( \int_{\frac{1}{n}}^1 s^{\alpha-1}(1-s)ds + \frac{1}{1-\Gamma}\int_0^1 A(t)dt \int_{\frac{1}{n}}^1 s^{\alpha-1}(1-s)ds \right)$$

$$= \left(\frac{L}{\beta}\right)^{q-1} \left( 1 + \frac{1}{1-\Gamma}\int_0^1 A(t)dt \right) \int_{\frac{1}{n}}^1 s^{\alpha-1}(1-s)ds$$

$$\le \left(\frac{L}{\beta}\right)^{q-1} \left( 1 + \frac{1}{1-\Gamma}\int_0^1 A(t)dt \right) \left( \frac{1}{\alpha} - \frac{1}{\alpha+1} \right)$$

$$< +\infty.$$

Hence, $T_n(\Omega)$ is bounded for $n = 3,4,...$ On the other hand, given $\varepsilon > 0$, let

$$\delta = \frac{\beta^{q-1}}{L^{q-1}\left(\frac{1}{\alpha-1} + \frac{1}{\alpha}\right)},$$

then, for each $\omega \in \Omega$, $t_1, t_2 \in [0,1]$, $t_1 < t_2$, and $t_2 - t_1 < \delta$, one has

$$|(T_n\omega)(t_2) - (T_n\omega)(t_1)| < \varepsilon.$$

That is to say that $T_n(\Omega)$ has equicontinuity. In fact, we consider three situations.

(1) $0 < t_1 < t_2 < \frac{1}{n}$.

$$|(T_n\omega)(t_2) - (T_n\omega)(t_1)|$$

$$= \left| \int_{\frac{1}{n}}^1 G(t_2,s)\varphi_q(\int_{\frac{1}{n}}^1 \tau^{\beta-1} f(\tau,-\omega(\tau)d\tau)ds - \int_{\frac{1}{n}}^1 G(t_1,s)\varphi_q(\int_{\frac{1}{n}}^1 \tau^{\beta-1} f(\tau,-\omega(\tau)d\tau)ds \right|$$

$$\leq \int_{\frac{1}{n}}^{1}\left|G(t_2,s)-G(t_1,s)\right|\varphi_q\left(\int_{\frac{1}{n}}^{1}\tau^{\beta-1}f(\tau,-\omega(\tau)d\tau\right)ds$$

$$\leq \left(\frac{L}{\beta}\right)^{q-1}\int_{\frac{1}{n}}^{1}\left|G(t_2,s)-G(t_1,s)\right|ds$$

$$\leq \left(\frac{L}{\beta}\right)^{q-1}\int_{\frac{1}{n}}^{1}(t_2-t_1)s^{\alpha-2}(1-s)ds$$

$$\leq \left(\frac{L}{\beta}\right)^{q-1}(t_2-t_1)\left(\frac{1}{\alpha-1}-\frac{1}{\alpha}\right)$$
$$<\varepsilon$$

(2) $0<t_1<\frac{1}{n}<t_2<1$

$$\left|(T_n\omega)(t_2)-(T_n\omega)(t_1)\right|$$

$$=\left|\int_{\frac{1}{n}}^{1}G(t_2,s)\varphi_q\left(\int_{\frac{1}{n}}^{1}\tau^{\beta-1}f(\tau,-\omega(\tau)d\tau\right)ds-\int_{\frac{1}{n}}^{1}G(t_1,s)\varphi_q\left(\int_{\frac{1}{n}}^{1}\tau^{\beta-1}f(\tau,-\omega(\tau)d\tau\right)ds\right|$$

$$\leq \int_{\frac{1}{n}}^{1}\left|G(t_2,s)-G(t_1,s)\right|\varphi_q\left(\int_{\frac{1}{n}}^{1}\tau^{\beta-1}f(\tau,-\omega(\tau)d\tau\right)ds$$

$$\leq \left(\frac{L}{\beta}\right)^{q-1}\int_{\frac{1}{n}}^{1}\left|G(t_2,s)-G(t_1,s)\right|ds$$

$$\leq \left(\frac{L}{\beta}\right)^{q-1}\left(\int_{\frac{1}{n}}^{t_2}\left|G(t_2,s)-G(t_1,s)\right|ds+\int_{t_2}^{1}\left|G(t_2,s)-G(t_1,s)\right|ds\right)$$

$$\leq \left(\frac{L}{\beta}\right)^{q-1}\left(\int_{\frac{1}{n}}^{t_2}s^{\alpha-2}[(s-t_1)+(t_2-t_1)]ds+\int_{t_2}^{1}(t_2-t_1)s^{\alpha-2}(1-s)ds\right)$$

$$\leq \left(\frac{L}{\beta}\right)^{q-1}(t_2-t_1)\int_0^1\left(s^{\alpha-2}+s^{\alpha-1}\right)ds$$
$$\leq \left(\frac{L}{\beta}\right)^{q-1}(t_2-t_1)\left(\frac{1}{\alpha-1}+\frac{1}{\alpha}\right)$$
$$<\varepsilon$$

(3) $\frac{1}{n}<t_1<t_2<1$

$$\left|(T_n\omega)(t_2)-(T_n\omega)(t_1)\right|$$

$$= \left| \int_{\frac{1}{n}}^{1} G(t_2,s)\varphi_q(\int_{\frac{1}{n}}^{1} \tau^{\beta-1} f(\tau,-\omega(\tau)d\tau)ds - \int_{\frac{1}{n}}^{1} G(t_1,s)\varphi_q(\int_{\frac{1}{n}}^{1} \tau^{\beta-1} f(\tau,-\omega(\tau)d\tau)ds \right|$$

$$\leq \int_{\frac{1}{n}}^{1} |G(t_2,s)-G(t_1,s)| \varphi_q(\int_{\frac{1}{n}}^{1} \tau^{\beta-1} f(\tau,-\omega(\tau)d\tau)ds$$

$$\leq \left(\frac{L}{\beta}\right)^{q-1} \int_{\frac{1}{n}}^{1} |G(t_2,s)-G(t_1,s)| ds$$

$$\leq \left(\frac{L}{\beta}\right)^{q-1} \left( \int_{\frac{1}{n}}^{t_1} |G(t_2,s)-G(t_1,s)| ds + \int_{t_1}^{t_2} |G(t_2,s)-G(t_1,s)| ds + \int_{t_2}^{1} |G(t_2,s)-G(t_1,s)| ds \right)$$

$$\leq \left(\frac{L}{\beta}\right)^{q-1} \left( \int_{\frac{1}{n}}^{t_1} (t_2-t_1) s^{\alpha-1} ds + \int_{t_1}^{t_2} (1-t_2)s^{\alpha-1} - t_1 s^{\alpha-2}(1-s)ds + \int_{t_2}^{1} (t_2-t_1) s^{\alpha-2}(1-s)ds \right)$$

$$\leq \left(\frac{L}{\beta}\right)^{q-1} (t_2-t_1) \int_0^1 \left(s^{\alpha-2}+s^{\alpha-1}\right) ds$$
$$\leq \left(\frac{L}{\beta}\right)^{q-1} (t_2-t_1) \left(\frac{1}{\alpha-1}+\frac{1}{\alpha}\right)$$
$$< \varepsilon$$

By the means of the Arzela-Ascoli theorem, we see that $T_n : P \to P$ are completely continuous operators.

Secondly, it is clear that $T : P \to P$. We prove that $T_n : P \to P$ uniformly converges to $T$ and $T : P \to P$ is completely continuous too. With the use of Lemma 2.9, we have

$$\varphi_q(A+B) < \varphi_q(A) + 2\varphi_q(B) + (q-1)(A+B)^{q-2} B$$

Given $\varepsilon > 0$ ,let

$$N = \left( \frac{(2+q)\left(\frac{L}{\beta}\right)^{q-1} H\left(1+\frac{1}{1-\Gamma}\int_0^1 A(t)dt\right)}{\varepsilon} \right)^{\alpha}$$

Then $\|T_n\omega - T\omega\| < \varepsilon,$ for all $n > N$ .In fact,

$$\|T_n\omega - T\omega\| = \max_{0\leq t\leq 1} |(T_n\omega)(t) - (T\omega)(t)|$$

$$\leq \left| \int_0^1 G(t,s)\varphi_q(\int_0^1 \tau^{\beta-1} f(\tau,-\omega(\tau))d\tau)ds - \int_{\frac{1}{n}}^{1} G(t,s)\varphi_q(\int_{\frac{1}{n}}^{1} \tau^{\beta-1} f(\tau,-\omega(\tau))d\tau)ds \right|$$

$$\leq \Big| \int_0^1 G(t,s)\Big[\varphi_q\Big(\int_{\frac{1}{n}}^1 \tau^{\beta-1} f(\tau,-\omega(\tau))d\tau\Big) + 2\varphi_q\Big(\int_0^{\frac{1}{n}} \tau^{\beta-1} f(\tau,-\omega(\tau))d\tau\Big)ds$$
$$+(q-1)\Big(\int_0^1 \tau^{\beta-1} f(\tau,-\omega(\tau))d\tau\Big)^{q-2}\int_0^{\frac{1}{n}} \tau^{\beta-1} f(\tau,-\omega(\tau))\Big]d\tau ds$$
$$-\int_{\frac{1}{n}}^1 G(t,s)\varphi_q\Big(\int_{\frac{1}{n}}^1 \tau^{\beta-1} f(\tau,-\omega(\tau))d\tau\Big)ds\Big|$$

$$\leq (2+q)\left(\frac{L}{\beta}\right)^{q-1}\left(\frac{1}{n}\right)^{\beta}\int_0^1 G(t,s)ds$$
$$= (2+q)\left(\frac{L}{\beta}\right)^{q-1}\left(\frac{1}{n}\right)^{\beta}\int_0^1\left(G_0(t,s)+\frac{1}{1-\Gamma}\int_0^1 G_0(t,s)A(t)dt\right)ds$$
$$\leq (2+q)\left(\frac{L}{\beta}\right)^{q-1}\left(\frac{1}{n}\right)^{\beta}\int_0^1\left(G_0(s,s)+\frac{1}{1-\Gamma}\int_0^1 A(t)dtG_0(s,s)\right)ds$$
$$\leq (2+q)\left(\frac{1}{n}\right)^{\beta}\left(\frac{L}{\beta}\right)^{q-1} H\left(1+\frac{1}{1-\Gamma}\int_0^1 A(t)dt\right)$$

$$<\varepsilon$$

By the use of Lemma2.10, $T: P \to P$ is completely continuous. □

Denote

$$M = \left(\left(\frac{1}{\beta}\right)^{q-1}\int_0^1\left(G_0(s,s)+\frac{1}{1-\Gamma}g_A(s)\right)ds\right)^{-1},$$

$$N = \left(\left(\frac{1}{\beta}\right)^{q-1}\int_{\frac{1}{4}}^{\frac{3}{4}}\left(G_0(s,s)+\frac{1}{1-\Gamma}g_A(s)\right)ds\right)^{-1}$$

**Theorem 3.1** Let $f(t,-\omega)$ be continuous on $[0,1]\times(-\infty,0]$. Assume that there exist two different positive constants $r_2$, $r_1$ and $r_2 \neq r_1$ such that

(HI) $f(t,-\omega(t)) \leq \varphi_p(Mr_1)$, for $(t,-\omega) \in [0,1]\times[-r_1,0]$;

(H2) $f(t,-\omega(t)) \geq \varphi_p(Nr_2)$, for $(t,-\omega) \in \left[\frac{1}{4},\frac{3}{4}\right]\times\left[-r_2,-\frac{1}{4}r_2\right]$.

Then Problem (1.1), (1.2),(1.3) has at least one positive solution $-\omega$ such that

$$\min\{r_1,r_2\} \leq \|\omega\| \leq \max\{r_1,r_2\}$$

*Proof* By Lemma 3.1, $T: P \to P$ is completely continuous. Without loss of generality, suppose $0 < r_1 < r_2$, and let

$$\Omega_1 := \{\omega \in P \mid \|\omega\| < r_1\}, \qquad \Omega_2 := \{u \in P \mid \|\omega\| < r_2\}.$$

For $\omega \in \partial\Omega_1$, we have $0 \le \omega(t) \le r_1$ for all $t \in [0,1]$. It follows from (H1) that

$$\begin{aligned}\|T\omega\| &= \max_{0\le t\le 1}\left|\int_0^1 G(t,s)\varphi_q(\int_0^1 \tau^{\beta-1} f(\tau,-\omega(\tau)d\tau)ds\right| \\ &\le Mr_1\left(\frac{1}{\beta}\right)^{q-1}\int_0^1\left(G_0(s,s)+\frac{1}{1-\Gamma}g_A(s)\right)ds = r_1 = \|\omega\|\end{aligned}$$

So , $\|T\omega\| \le \|\omega\|$ ,For $\omega \in \partial\Omega_1$.

For $\omega \in \partial\Omega_2$ ,by the definition of $P$ ,we have

$$\omega(t) \le \frac{1}{4}\|\omega\| = \frac{1}{4}r_2, \qquad t \in \left[\frac{1}{4},\frac{3}{4}\right].$$

By assumption (H1), for $t \in \left[\frac{1}{4},\frac{3}{4}\right]$, we have

$$\begin{aligned}(T\omega)(t) &= \int_0^1 G(t,s)\varphi_q(\int_0^1 \tau^{\beta-1} f(\tau,-\omega(\tau)d\tau)ds \\ &\ge \int_0^1 \frac{1}{4}\left(G_0(s,s)+\frac{1}{1-\Gamma}g_A(s)\right)\varphi_q(\int_0^1 \tau^{\beta-1} f(\tau,-\omega(\tau)d\tau)ds \\ &\ge Nr_2\frac{1}{\beta}\int_{\frac{1}{4}}^{\frac{3}{4}}\frac{1}{4}\left(G_0(s,s)+\frac{1}{1-\Gamma}g_A(s)\right)ds = r_2 = \|\omega\|\end{aligned}$$

So , $\|T\omega\| \ge \|\omega\|$ ,For $\omega \in \partial\Omega_2$ .

Therefore, by Lemma2.11, we complete the proof. □

## 4. Multiplicity

**Theorem 4.1** Suppose $f(t,-\omega)$ is continuous on $[0,1]\times(-\infty,0]$ and there exist constants $0 < a < \frac{1}{4}b$ such that the following assumptions hold:

(A1) $f(t,-\omega) \le \varphi_p(Ma)$, for $(t,-\omega) \in [0,1]\times[-a,0]$;

(A2) $f(t,-\omega) \ge \varphi_p\left(\frac{1}{4}Nb\right)$, for $(t,-\omega) \in \left[\frac{1}{4},\frac{3}{4}\right]\times\left[-b,-\frac{1}{4}b\right]$;

(A3) $f(t,-\omega) \le \varphi_p(Mb)$, for $(t,-\omega) \in [0,1]\times[-b,0]$.

Then the boundary value problem (1.1),(1.2),(1.3) has at least three negative solutions $-\omega_1,-\omega_2,-\omega_3$ with

$$\max_{0\le t\le 1}|-\omega_1(t)| < a,\ \ \frac{1}{4}b < \min_{\frac{1}{4}\le t\le\frac{3}{4}}|-\omega_2(t)|,\ \ a < \max_{0\le t\le 1}|-\omega_3(t)|,\ \ \min_{\frac{1}{4}\le t\le\frac{3}{4}}|-\omega_3(t)| < \frac{1}{4}b$$

The proof of Theorem 4.1 is similar to that in [18],so we omit it.

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