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Mean Value Theorems in q,ω-Calculus

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Abstract: In this paper, some properties of continuous functions in q,ω-analysis are investigated. The behavior of q,ω-derivative in a neighborhood of a local extreme point is described. Two theorems are proved which are q,ω-analogons of the fundamental theorems of the differential calculus. Also, two *q*-integral mean value theorems are proved.

1. Introduction

Quantum difference operator are receiving an interest role due to their applications in several mathematical areas,see,e.g.[1,2,3].In [4], Hahn introduced the quantum difference operator $D_{q,\omega}$, where $q \in]0,1[$ and $\omega > 0$ are fixed. The Hahn operator unifies (in the limit) the two most well known and used quantum difference operators: the Jackson *q*-difference derivative D_q, where $q \in]0,1[$; and the forward difference Δω, where $\omega > 0$.The Hahn difference operator is a successful tool for constructing families of orthogonal polynomials and investigating some approximation problems . However, only in 2009 the construction of a proper inverse of $D_{q,\omega}$ and the associated integral calculus was given .

The plan of the paper is as follows. In Sect 2,some properties of continuous functions in q,ω-analysis are investigated. The behavior of q,ω-derivative in a neighborhood of a local extreme point is described. Two theorems are proved which are q,ω-analogons of the fundamental theorems of the differential calculus. In Sect 3,two q,ω-integral mean value theorems are proved .

Let $q \in]0,1[$ and $\omega \in]0,+\infty[$ be given. Define $\omega_0 = \frac{\omega}{1-q}$.Throughout all the paper we assume I to be an interval of R containing ω_0. A q,ω-natural number $[k]_{q,\omega} := \frac{\omega(1-q^k)}{1-q}$ for $k \in N_0 = N \cup \{0\}$.

Definition 1.1([5])(Hahn's difference operator). Let $f : I \to R$,The Hahn difference operator is defined by

$$D_{q,\omega} f(t) = \begin{cases} \dfrac{f(qt+\omega) - f(t)}{qt+\omega-t} & if \ \ t \neq \omega_0 \\ f'(t) & if \ \ t = \omega_0 \end{cases}$$

Provided that f is differentiable at ω_0 (we have using $f'(t)$ to denote the Frechet derivative).In this case we call $D_{q,\omega} f$ the q,ω-derivative of f and say that f is q,ω-derivatiable on I .

Definition 1.2 .([5])We define the fractional q,ω -derivative by

$$ {}_aD^{\alpha}_{q,\omega} f(x) = \begin{cases} ({}_aI^{-\alpha}_{q,\omega}) f(x), \alpha < 0 \\ f(x), \alpha = 0 \\ (D^{\lceil\alpha\rceil}_{q,\omega}\, {}_aI^{\lceil\alpha\rceil-\alpha}_{q,\omega}) f(x), \alpha > 0 \end{cases} $$

Where $\lceil\alpha\rceil$ denotes the smallest integer greater or equal to α

2. Extreme values and q,ω -derivative

We will consider relations between, on one side, extreme value of a continuous function and, on the other side, derivatives and q,ω -derivatives.

THEOREM 2.1. ([6]) Let $f(x)$ be a continuous function on a segment $[a,b]$ and let $c \in (a,b)$ be a point of its local maximum.

(i) If $0 < a < b$, then there exists $\hat{q} \in (0,1)$ such that

$$\forall q \in (\hat{q},1), \omega < c - qc, \qquad D_{q,\omega} f(c) \geq 0 \tag{2.1.1}$$

$$\forall q \in (\hat{q},1), \omega > c - qc \quad \text{or} \quad \forall q \in (1,\hat{q}^{-1}), \omega > c - qc, \qquad D_{q,\omega} f(c) \leq 0 \tag{2.1.2}$$

(ii) If $a < b < 0$, then there exists $\hat{q} \in (0,1)$ such that

$$\forall q \in (\hat{q},1), \omega < c - qc, \qquad D_{q,\omega} f(c) \leq 0 \tag{2.1.3}$$

$$\forall q \in (\hat{q},1), \omega > c - qc \quad \text{or} \quad \forall q \in (1,\hat{q}^{-1}), \omega > c - qc, \quad D_{q,\omega} f(c) \geq 0 \tag{2.1.4}$$

Furthermore, if $0 < a < b$,then

$$(\forall q \in (\hat{q},1), \omega < c - qc) \cup (\forall q \in (1,\hat{q}^{-1}), \omega > c - qc)\, (\exists \xi \in (a,b))\ D_{q,\omega} f(\xi) = 0. \tag{2.1.5}$$

Proof. Since the proofs of (i) and (ii) are very similar, we will expose only the first one.

Now we prove (2.1.1). Since *c* is a point of local maximum of the function $f(x)$, there exists $\varepsilon > 0$, such that $f(x) \leq f(c)$, for all $x \in (c-\varepsilon, c+\varepsilon) \subset (a,b)$. Let $q_0 \in (0,1)$ such that $c - \varepsilon < q_0 c <$ c. Now, for all $q \in (q_0,1)$, $\omega < c - cq$, it is valid $qc + \omega < c$ and $f(qc+\omega) \leq f(c)$, wherefrom $D_{q,\omega} f(c) \geq 0$.

Then we prove (2.1.2). When $q \in (q_0,1)$, $\omega > c - cq$,it is valid $qc + \omega > c$ and $f(qc+\omega) \leq f(c)$, wherefrom $D_{q,\omega} f(c) \leq 0$. In a similar way, for the above $\varepsilon > 0$, there exists $q_1 \in (0,1)$ such that

$c < c/q_1 < c + \varepsilon$ and for all $q \in (1, q_1^{-1}), \omega \in]0, +\infty[$,then

$c < qc < qc + \omega$, it holds $D_{q,\omega} f(c) \le 0$.

We may choose $\hat{q}$, st $\max\{q_0, q_1\} < \hat{q} < 1$ and $c/\hat{q} \le \omega_0$. since $\omega_0 > c$ .Next we consider (2.15). Let $q \in (\hat{q}, 1)$ be an arbitrary real number. Then $\eta = (c-\omega)/q < \omega_0$, wherefrom $f(c) \ge f(\eta)$, since $c = q\eta + \omega$ *a.e.* $f(q\eta + \omega) \ge f(\eta)$. From $q\eta + \omega < \eta$ we conclude $D_{q,\omega} f(\eta) \ge 0$ .As $f(x)$ is a continuous function, $D_{q,\omega} f(x)$ is continuous in (a,b) too. Since $D_{q,\omega} f(c) \le 0$, $D_{q,\omega} f(\eta) \ge 0$, where $c, \eta \in (a,b)$, there exists $\xi \in (\eta, c) \subset (a,b)$, such that $D_{q,\omega} f(\xi) = 0$. □

Analogously, for an arbitrary $q \in (1, \hat{q}^{-1})$, $c < \omega_0$ the number $\eta = (c-\omega)/q < c$, wherefrom $D_{q,\omega} f(\eta) \ge 0$. Since $D_{q,\omega} f(c) \le 0$,we have proved the existence of a zero of $D_{q,\omega} f(x)$ for $q \in (1, \hat{q}^{-1})$.

In order to better understand the proof process of one case of (1.1.2) , we cite an example.

EXAMPLE 2.1. Let us consider $f(x) = (x-1)(3-x) + 2$. Its maximum is at $c = 2$, but q, ω -derivative is $(D_{q,\omega} f)(x) = -x[2]_q - \omega + 4$ and it vanishes at the point $\xi = (4-\omega)/(1+q)$. Since $\hat{q} \in (0,1)$, $q \in (\hat{q}, 1)$,we can let $\hat{q} = \frac{1}{2}$, $q = \frac{2}{3}$ .since $\omega_0 > c$,then we have $\omega > (1-q)c$, so we can let $\omega = \frac{5}{6}$. From the above proof $\eta = (c-\omega)/q = \frac{7}{4}$.

$\xi = (4-\omega)/(1+q) = \frac{19}{10}$. Obviously,it satisfied $\xi \in (\eta, c)$. We have

$$D_{\frac{2}{3},\frac{5}{6}} f(c) = D_{\frac{2}{3},\frac{5}{6}} f(2) = -2[2]_{\frac{2}{3}} - \frac{5}{6} + 4 = -\frac{1}{6} < 0,$$

$$D_{\frac{2}{3},\frac{5}{6}} f(\eta) = D_{\frac{2}{3},\frac{5}{6}} f(\frac{7}{4}) = -\frac{7}{4}[2]_{\frac{2}{3}} - \frac{5}{6} + 4 = \frac{1}{4} > 0,$$

$$D_{\frac{2}{3},\frac{5}{6}} f(\xi) = D_{\frac{2}{3},\frac{5}{6}} f(\frac{19}{10}) = -\frac{19}{10}[2]_{\frac{2}{3}} - \frac{5}{6} + 4 = 0$$

In above example we can easy understand one case of (2.1.2). In a similar way, we can prove the next theorem.

THEOREM 2.2.([6]) Let $f(x)$ be a continuous function on a segment $[a,b]$ and let $c \in (a,b)$ be a point of its local minimum.

(i) If $0 < a < b$, then there exists $\hat{q} \in (0,1)$ such that

$$\forall q \in (\hat{q},1), \omega < c - qc, D_{q,\omega} f(c) \le 0 \qquad (2.2.1)$$

$$\forall q \in (\hat{q},1), \omega > c - qc \text{ or } \forall q \in (1,\hat{q}^{-1}), \omega > c - qc, \qquad D_{q,\omega} f(c) \ge 0 \qquad (2.2.2)$$

(ii) If $a < b < 0$, then there exists $\hat{q} \in (0,1)$ such that

$$\forall q \in (\hat{q},1), \omega < c - qc, D_{q,\omega} f(c) \ge 0 \qquad (2.2.3)$$

$$\forall q \in (\hat{q},1), \omega > c - qc \text{ or } \forall q \in (1,\hat{q}^{-1}), \omega > c - qc, \quad D_{q,\omega} f(c) \le 0 \qquad (2.2.4)$$

Moreover, $(\forall q \in (\hat{q},1), \omega < c - qc) \cup (\forall q \in (1,\hat{q}^{-1}), \omega > c - qc)\ (\exists \xi \in (a,b))\ D_{q,\omega} f(\xi) = 0$.

REMARK. If $f(x)$ is differentiable for all $x \in (a,b)$, then $\lim_{q\uparrow 1, \omega \downarrow 0} D_{q,\omega} f(x) = f'(x)$. So, if $c \in (a,b)$ is a point of local extreme of $f(x)$, we have $f'(c) = D_{1,0} f(c) = D_1 f(c) = 0$.

3. Some q,ω-mean value theorems

By using the previous results, we can establish and prove analogons of well known mean value theorems in q,ω-calculus.

THEOREM 3.1. (q,ω-Rolle) Let $f(x)$ be a continuous function on $[a,b]$ satisfying $f(a) = f(b)$. Then there exists $\hat{q} \in (0,1)$, such that

$$(\forall q \in (\hat{q},1), \omega < c - qc) \cup (\forall q \in (1,\hat{q}^{-1}), \omega > c - qc)\ (\exists \xi \in (a,b))\ D_{q,\omega} f(\xi) = 0$$

Proof. If $f(x)$ is not a constant function on $[a,b]$, then it attains its extreme value in some point in (a,b). But, according to Theorems 2.1-2.2, $D_{q,\omega} f(x) = 0$ vanishes at a point $\xi \in (a,b)$.

THEOREM 3.2 (q,ω-Lagrange). Let $f(x)$ be a continuous function on $[a,b]$. Then there exists $\hat{q} \in (0,1)$, such that

$$(\forall q \in (\hat{q},1), \omega < c - qc) \cup (\forall q \in (1,\hat{q}^{-1}), \omega > c - qc)\ (\exists \xi \in (a,b))\ f(b) - f(a) = D_{q,\omega} f(\xi)(b-a)$$

Proof. The statement follows by applying the previous theorem to the function

$f(x)-x(f(b)-f(a))/(b-a)$.

4. Mean value theorems for q,ω-integrals

Definition 4.1. Let I be a closed interval of R such that $\omega_0, a,b \in I$ .For $f:I\to R$ we defined the q,ω-integral of f from a to b by

$$\int_a^b f(t)d_{q,\omega}t := \int_{\omega_0}^b f(t)d_{q,\omega}t - \int_{\omega_0}^a f(t)d_{q,\omega}t \quad (1)$$

$$\int_{\omega_0}^x f(t)d_{q,\omega}t := (x(1-q)-\omega)\sum_{k=0}^{\infty} q^k f(xq^k+[k]_{q,\omega}) \quad (2)$$

With $[k]_{q,\omega} := \frac{(1-q^k)\omega}{1-q}$ for $k\in N_0 = N\cup\{0\}$,provided that the series converges at $x=a$ and $x=b$.In this case, f is called q,ω-integral over $[a,b]$,for all $a,b\in I$.

Note that, in the integral formulas (1),(2),when $\omega\downarrow 0$ we obtain the Jackson *q*-integral

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt$$

Where

$$\int_a^x f(t)d_qt = x(1-q)\sum_{k=0}^{\infty} q^k f(xq^k)$$

Notice that $I(f)=\int_a^b f(t)dt = \lim_{q\uparrow 1} I_q f(t)$

THEOREM 4.1. Let $f(x)$ be a continuous function on a segment $[\omega_0,b]$, $(b>\omega_0)$, $\omega_0 = \frac{\omega}{1-q}$.

Then

$$(\forall q\in(0,1),\omega<b-bq)(\exists\xi\in[\omega_0,b]): I_{q,\omega}(f)=\int_{\omega_0}^b f(t)d_{q,\omega}(t)=(b-\omega_0)f(\xi)$$

Proof. Since $f(x)$ is a continuous function on the segment $[\omega_0,b]$, it attains its minimum m and maximum M and takes all values between. According to assumption $0<q<1$, $\omega<b-bq$,we have $\omega_0 \le bq^k+[k]_{q,\omega}\le b$ and $m\le f(bq^k+[k]_{q,\omega})\le M$. Multiply both sides of this eqution by a number that is greater than zero .we obtain

$$(b(1-q)-\omega)\sum_{k=0}^{\infty} q^k m < (b(1-q)-\omega)\sum_{k=0}^{\infty} q^k f(bq^k+[k]_{q,\omega}) < (b(1-q)-\omega)\sum_{k=0}^{\infty} q^k M$$ By

definition 4.1(2),we know $I_{q,\omega}(f)=(b(1-q)-\omega)\sum_{k=0}^{\infty}q^k f(bq^k+[k]_{q,\omega})$ .Now,the equality is going to be

$$(b(1-q)-\omega)\sum_{k=0}^{\infty}q^k m \le I_{q,\omega}(f) \le (b(1-q)-\omega)\sum_{k=0}^{\infty}q^k M$$

By $\omega_0=\frac{\omega}{1-q}$,we have $(b-\omega_0)\ m \le I_{q,\omega}(f) \le (b-\omega_0)\ M$

That is $m \le \frac{1}{(b-\omega_0)} I_{q,\omega}(f) \le M$ .we use the intermediate value theorem for continuous function.

There exists $\xi \in [\omega_0, b]$ such that $f(\xi)=\frac{1}{(b-\omega_0)} I_{q,\omega}(f)$ .Then

$$I_{q,\omega}(f)=(b-\omega_0)f(\xi) \qquad \square$$

Moreover, if we define $\int_a^b f(t)d_{q,\omega}t=\int_{\omega_0}^b f(t)d_{q,\omega}t-\int_{\omega_0}^a f(t)d_{q,\omega}t$ then the next theorem is valid.

THEOREM 4.2. Let $f(x)$ be a continuous function on a segment $[a,b]$. Then there exists $\hat{q}\in(0,1)$ $\overline{\omega}\in(0,1)$ such that

$$(\forall q\in(\hat{q},1),\omega\in(0,\overline{\omega}))(\exists\xi\in[a,b]): I_{q,\omega}(f)=\int_a^b f(t)d_{q,\omega}(t)=(b-a)f(\xi)$$

Proof. It is easy to prove that $I(f)=\int_a^b f(t)dt=\lim_{q\uparrow 1} I_q f(t)=\lim_{q\uparrow 1,\omega\downarrow 0} I_{q,\omega} f(t)$, i.e.

$$(\forall\varepsilon>0)(\exists q_0\in(0,1),\forall q\in(q_0,1),\omega\in(0,\overline{\omega})): I(f)-\varepsilon<I_{q,\omega}(f)<I(f)+\varepsilon$$

According to the well known mean value theorem for integrals, we have

$$(\exists c\in(a,b)) : I(f)=f(c)(b-a)$$

Let $\varepsilon<(b-a)\min\{M-f(c), f(c)-m\}$, where m and M are the minimum and maximum of $f(x)$ on $[a,b]$. Now,

$$\exists\ \hat{q}\in(0,1)\ (\forall q\in(\hat{q},1),\omega\in(0,\overline{\omega})): f(c)-\frac{\varepsilon}{b-a}<\frac{1}{b-a}I_{q,\omega}(f)<f(c)+\frac{\varepsilon}{b-a}$$

hence $m\le\frac{1}{b-a}I_{q,\omega}(f)\le M$. Since $f(x)$ is a continuous function on the segment $[a,b]$, it takes all values between m and M, i.e.

$$(\exists\xi \in (a,b)) : \frac{1}{b-a} I_{q,\omega}(f) = f(\xi)$$

what we wanted to prove. □

THEOREM 4.3. Let $f(x)$ and $g(x)$ be some continuous functions on a segment $[a,b]$. Then there exists $\hat{q} \in (0,1)$, $\overline{\omega} \in (0,1)$ such that

$$(\forall q \in (\hat{q},1), \omega \in (0,\overline{\omega}))(\exists\xi \in (a,b)) : I_{q,\omega}(fg) = g(\xi) I_{q,\omega}(f)$$

Proof. According to the second mean value theorem for integrals, we have

$$(\exists c \in (a,b)) : I(fg) < g(c) I(f)$$

Hence $\lim\limits_{q\uparrow 1,\omega\downarrow 0} I_{q,\omega}(fg) = I(fg) = g(c)I(f) = g(c) \lim\limits_{q\uparrow 1,\omega\downarrow 0} I_{q,\omega}(f)$, i.e. $\lim\limits_{q\uparrow 1,\omega\downarrow 0} \frac{I_{q,\omega}(fg)}{I_{q,\omega}(f)} = g(c)$

Now,

$$(\exists q_0 \in (0,1), \forall q \in (q_0,1), \omega \in (0,\overline{\omega})) : g(c) - \varepsilon < \frac{I_{q,\omega}(fg)}{I_{q,\omega}(f)} < g(c) + \varepsilon$$

Since $g(x)$ is a continuous function on the segment $[a,b]$ it attains its minimum m_g and maximum M_g.

Let $\varepsilon \le \min\{ M_g - g(c),\ g(c) - m_g \}$ .Hence

$$(\exists \hat{q} \in (0,1), \forall q \in (\hat{q},1) \omega \in (0,\overline{\omega})) : m_g < \frac{I_{q,\omega}(fg)}{I_{q,\omega}(f)} < M_g$$

Since $f(x)$ takes all values between m_g and M_g , we conclude that

$$(\exists\xi \in (a,b)) : \frac{I_{q,\omega}(fg)}{I_{q,\omega}(f)} = g(\xi)\,.$$ □

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