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Existence of positive solutions for second order nonlinear $q.\omega$- difference equations

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Abstract: We consider the existence of positive solutions for boundary value problems of a class of nonlinear $q.\omega$- difference equations . Firstly, analysis some properties of the Green function. The second, by applying a fixed point theorem in cones we investigate the existence of positive solutions for the boundary value problems .
Keywords: $q.\omega$- difference equations; boundary value problem; cones; existence of solutions.

1. Introduction

In 1949 Hahn [1] introduced the $q.\omega$-difference operator $D_{q.\omega}$, say Hahn's difference operator. It has been applied successfully in construction of families of orthogonal polynomials as well as in approximation problem [2-4]. However, during 68 years, few authors have studied Hahn's quantum calculus. We refer reader to the monographs of Aldwoah [5] [6], Bangerezako [7] and Artur M. C Brito da cru Z [8].

In this paper, we consider the boundary value problem of following $q.\omega$- difference equation .

$$(D^2_{q,\omega}x)(t) = -f(t, x(t)), \quad \omega_0 < t < b, \tag{1}$$

$$x(\omega_0) = x(b) = 0. \tag{2}$$

By using both the fixed point theorem, we show that existence of the positive solution for the problem (1) (2).

2. Preliminaries

Let $q \in (0,1)$ and $\omega > 0$. We introduce the real number $\omega_0 = \dfrac{\omega}{1-q}$.Let I be a real interval containing ω_0. For a function f defined on I, *Hahn's* difference operator $D_{q.\omega} f$ is given by

$$\left(D_{q.\omega} f\right)(x) = \frac{f(qx+\omega) - f(x)}{(q-1)x+\omega}, x \neq \omega_0, \ \left(D_{q.\omega} f\right)(\omega_0) = f'(\omega_0).$$

Definition 1 [7]. Let $a, b \in I$ and $a < b$. For $f : I \to \mathbb{R}$ the $q.\omega$-integrals is defined by

$$\int_a^b f(x) d_{q.\omega} x = \int_{\omega_0}^b f(x) d_{q.\omega} x - \int_{\omega_0}^a f(x) d_{q.\omega} x,$$

$$\int_{\omega_0}^x f(t) d_{q.\omega} t = (x(1-q) - \omega) \sum_{n=0}^{\infty} q^n f(xq^n + \omega [n]_q) \tag{3}$$

Provided that the series converges at $x = a, x = b.$ In that case, f is called $q.\omega$ integrable on $[a,b]$.

We say that f is $q.\omega$ integrable over I if it is $q.\omega$ integrable on $[a,b]$ for all $a,b \in \mathrm{I}$.

Lemma 1. Assume that $f:\mathrm{I}\to\square$ is continuous at ω_0 and,for each $x\in\mathrm{I}$,define $F(x) := \int_{\omega_0}^{x} f(t)d_{q.\omega}$,then F is continuous at ω_0 and $D_{q.\omega}F$ exists for every $x\in\mathrm{I}$ with $D_{q.\omega}F(x) = f(x).$ Conversely $\int_a^b (D_{q.\omega}f)(x)d_{q.\omega}x = F(b) - F(a), \forall a,b \in \mathrm{I}.$

Lemma 2. Assume that $f:\mathrm{I}\to\square$ is continuous at ω_0 and,for each $x\in\mathrm{I}$, f is $q.\omega$ integrable, then

$$\int_{\omega_0}^{t}\int_{\omega_0}^{s} f(\tau)d_{q.\omega}\tau\, d_{q.\omega}s = \int_{\omega_0}^{t}(t-\tau)f(\tau)d_{q.\omega} \qquad (4)$$

Proof. By Definition 1, we have

$$\int_{\omega_0}^{t}\int_{\omega_0}^{s} f(\tau)d_{q.\omega}\tau = (t(1-q)-\omega)\sum_{k=0}^{\infty} q^k \int_{\omega_0}^{q^k t+\omega[k]_q} f(\tau)d_{q.\omega}\tau$$

$$=(t(1-q)-\omega)\sum_{k=0}^{\infty} q^k((q^k t+\omega[k]_q)(1-q)-\omega)\sum_{m=0}^{\infty} q^m f((q^k t+\omega[k]_q)q^m+\omega[m]_q)$$

$$=(t(1-q)-\omega)\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} q^m q^k((q^k t+\omega[k]_q)(1-q)-\omega) f(q^{k+m}t+\omega[k+m]_q)$$

$$=(t(1-q)-\omega)\sum_{k=0}^{\infty}\sum_{m=k}^{\infty} q^m((q^k t+\omega[k]_q)(1-q)-\omega) f(q^m t+\omega[m]_q)$$

$$=(t(1-q)-\omega)\sum_{m=0}^{\infty}\sum_{k=0}^{m} q^m((q^k t+\omega[k]_q)(1-q)-\omega) f(q^m t+\omega[m]_q)$$

$$=(t(1-q)-\omega)\sum_{m=0}^{\infty} q^m f(q^m t+\omega[m]_q)\sum_{k=0}^{m}((q^k t+\omega[\mathrm{k}]_q)(1-q)-\omega)$$

$$=(t(1-q)-\omega)^2\sum_{m=0}^{\infty} q^m f(q^m t+\omega[m]_q)[m]_q$$

$$=(t(1-q)-\omega)\sum_{m=0}^{\infty} q^m(t-(tq^m+\omega[m]_q)) f(q^m t+\omega[m]_q)$$

$$= \int_{\omega_0}^{t} (t-\tau) f(\tau) d_{q.\omega}\tau.$$

Therefore, (4) holds.

Lemma 3. Assume that $f : \mathrm{I} \times \mathbb{R} \to \mathbb{R}$ be given. The unique solution of the problem(1) (2) is the function

$$x(t) = \int_{\omega_0}^{b} G(t,s) f(s, x(s)) d_{q.\omega}s.$$

where

$$G(t,s) = \frac{1}{b-\omega_0} \begin{cases} (b-s)(t-\omega) - (t-s)(b-\omega), \omega_0 \le s \le t \le b, \\ (b-s)(t-\omega), \omega_0 \le t \le s \le b. \end{cases}$$

Lemma 4. Let $G(t,s)$ be Green's function given in the statement of Lemma 3, then $G(t,s)$ satisfies the following conditions:

(i) $G(t,s) \ge 0, G(t,s) \le G(s,s), (t,s) \in [\omega_0, b]^2,$

(ii) There existe $\gamma \in (0,1)$ such that $\min_{t \in [\omega_0 + \frac{b-\omega_0}{4}, \omega_0 + \frac{3(b-\omega_0)}{4}]} G(t,s) \ge \gamma G(s,s).$

Proof. Let $t \in [\omega_0 + \frac{b-\omega_0}{4}, \omega_0 + \frac{3(b-\omega_0)}{4}]$, if $t < s$ then

$$\frac{G(t,s)}{G(s,s)} = \frac{t-\omega}{s-\omega} \ge \frac{\omega_0 + \frac{b-\omega_0}{4} - \omega}{b-\omega}.$$

If $s \le t,$ then

$$\frac{G(t,s)}{G(s,s)} = \frac{(b-s)(t-\omega)-(t-s)(b-\omega)}{(b-s)(s-\omega)} = \frac{t-\omega}{s-\omega} - \frac{(t-s)(\mathrm{b}-\omega)}{(b-s)(s-\omega)}$$

$$= \frac{t-\omega}{\mathrm{s}-\omega}(1 - \frac{(t-s)(\mathrm{b}-\omega)}{(b-s)(\mathrm{t}-\omega)}).$$

Since $\frac{(t-s)(b-\omega)}{(b-s)(t-\omega)} \le \frac{(\omega_0 + \frac{3}{4}(b-\omega_0) - s)(b-\omega)}{(b-s)(\omega_0 + \frac{3}{4}(b-\omega_0) - \omega)} \le \frac{3}{4} \frac{b-\omega}{\omega_0 + \frac{3}{4}(b-\omega_0) - \omega},$

we see that

$$\frac{G(t,s)}{G(s,s)} = 1 - \frac{3}{4} \frac{b-\omega}{\omega_0 + \frac{3(b-\omega_0)}{4} - \omega}.$$

Choose

$$\gamma = \min\{\frac{\omega_0 + \frac{b-\omega_0}{4} - \omega}{b-\omega}, 1 - \frac{3}{4}\frac{b-\omega}{\omega_0 + \frac{3(b-\omega_0)}{4} - \omega}\}.$$

Thus Lemma 4 holds.

Lemma 5. Let B be a Banach space and $P \subseteq \mathrm{B}$ be a cone in . Assume Ω_1 and Ω_2 are open subsets of B with $0 \in \Omega_1, \overline{\Omega}_1 \subseteq \Omega_2$, and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathbb{R}$ is completely continuous operator such that , ether

(i) $\|Ty\| \le \|y\|, y \in P \cap \partial\Omega_1, \|Ty\| \ge \|y\|, y \in P \cap \partial\Omega_2$ or

(ii) $\|Ty\| \ge \|y\|, y \in P \cap \partial\Omega_1, \|Ty\| \le \|y\|, y \in P \cap \partial\Omega_2$

Then T has a fixed point in $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. The main results

In this section, we give the existence of positive solutions of problem (1) (2). We notice that x solves (1) (2) if and only if x is a fixed point of the operator

$$Tx(t) = \int_{\omega_0}^{b} G(t,s) f(s, x(s)) d_{q.\omega} s. \qquad (5)$$

Where G is Green's function derived in Lemma 3 and $T : \mathrm{B} \to \mathrm{B}$, where B is the Banach space consisting of all maps $[\omega_0, b] \to \mathbb{R}$ when equipped with the usual supremum norm $\|\cdot\|$.

Let us also make the following declarations, which will be use in the sequel.

$$\eta = \frac{1}{\int_{\omega_0}^{b} G(s,s)\,\mathrm{d}_{q.\omega} s}, \qquad \lambda = \frac{1}{\int_{\omega_0+\frac{b-\omega_0}{4}}^{\omega_0+\frac{3(b-\omega_0)}{4}} \gamma G(\omega_0 + \frac{b-\omega_0}{2}, s)\,\mathrm{d}_{q.\omega} s}.$$

Let us also introduce two conditions on the behaviour of f that will be useful in the sequel. These are standard assumptions on the growth of the non-linearity f.

(C_1) There exists a number $r > 0$ such that $f(t,x) \le \eta r$, whenever $0 \le x \le r$.

(C_2) There exists a number $r > 0$ such that $f(t,x) \ge \lambda r$, whenever $\gamma r \le x \le r$.

Where γ is the constant deduced in Lemma 4.

We now can prove the following existence result.

Theorem 1. Suppose that there are distinct $r_1, r_2 > 0$ such that condition (C_1) holds at $r = r_1$ and condition (C_2) holds at $r = r_2$. Suppose also that $f(t,x) \ge 0$ and continuous. Then the boundary value problem (1) (2) has at least one positive solution, say x_0, such that $\|x_0\|$ lies between r_1 and r_2.

Proof. We shall assume without loss of generality that $0 < r_1 < r_2$, consider the set

$$\mathrm{K} = \{x \in \mathrm{B} : x(t) \geq 0, \min x(t) \geq \gamma \|x\|\},$$

which is a cone with $\mathrm{K} \subseteq \mathrm{B}$ .Observe that $T : \mathrm{K} \to \mathrm{K}$, for we observe that

$$\min_{t \in [\omega_0 + \frac{b-\omega_0}{4}, \omega_0 + \frac{3(b-\omega_0)}{4}]} (Tx)(t) \geq \gamma \int_{\omega_0}^{b} G(s,s)\, \mathrm{f}(\mathrm{s}, \mathrm{x}(\mathrm{s}))\, \mathrm{d}_{q.\omega} s = \gamma \|Tx\|,$$

whence $Tx \in \mathrm{K}$, as claimed. Also, it is easy to see that T is a completely continuous operator. Now, put $\Omega_1 = \{x \in \mathrm{K}, \|x\| < r_1\}$. Note that for $x \in \partial\Omega_1$,we have that $\|x\| = r_1$ so that condition (C_1) holds for all $x \in \partial\Omega_1$. Then for $x \in \mathrm{K} \cap \partial\Omega_1$,we find

$$\begin{aligned} \|Tx\| &= \max_{t \in [\omega_0, b]} \int_{\omega_0}^{b} G(t,s) f(s, x(s)) d_{q.\omega} s \\ &\leq \int_{\omega_0}^{b} G(s,s) f(s, x(s)) d_{q.\omega} s \\ &\leq \eta r_1 \int_{\omega_0}^{b} G(s,s) f(s, x(s)) d_{q.\omega} s = r_1 = \|x\|, \end{aligned}$$

whence we find that $\|Tx\| \leq \|x\|$. Thus T is a cone compression on $\mathrm{K} \cap \partial\Omega_1$.

Next, put $\Omega_2 = \{x \in \mathrm{K}, \|x\| < r_2\}$. Note that for $x \in \partial\Omega_2$,we have that $\|x\| = r_2$ so that condition (C_2) holds for all $x \in \partial\Omega_2$. Then for $x \in \mathrm{K} \cap \partial\Omega_2$, we find

$$\begin{aligned} Tx(\omega_0 + \frac{b-\omega_0}{2}) &= \int_{\omega_0}^{b} G(\omega_0 + \frac{b-\omega_0}{2}, s) f(s, x(s)) d_{q.\omega} s \\ &\geq \int_{\omega_0 + \frac{b-\omega_0}{4}}^{\omega_0 + \frac{3(b-\omega_0)}{4}} \gamma G(s,s) f(s, x(s)) d_{q.\omega} \\ &\geq \lambda r_2 \int_{\omega_0 + \frac{b-\omega_0}{4}}^{\omega_0 + \frac{3(b-\omega_0)}{4}} \gamma G(\omega_0 + \frac{b-\omega_0}{2}, s) d_{q.\omega} s = r_2, \end{aligned}$$

whence we find that $\|Tx\| \geq \|x\|$. Thus T is a cone expansion on $\mathrm{K} \cap \partial\Omega_2$. So, it follows by Lemma 5 that the operator has fixed T point. It means that (1) (2) has a positive solution with $r_1 \leq x_0 \leq r_2$.

Theorem 2. Assume that there exists a constant $M > 0$ such that

$$\max_{(t,x)\in[\omega_0,b]\times[-M,M]} |f(t,x)| \le \frac{M}{\int_{\omega_0}^{b} G(s,s)d_{q.\omega}s}.$$

Then (1) (2) has a solution, say $x_0(t)$ such that $|x_0(t)| \le M$, for each $t \in [\omega_0, b]$.

Proof. Let T be the operator defined by (5). Denote

$\mathrm{B}_M = \{x : [\omega_0, b] \to \mathbb{R}, \|x\| \le M\}$, observe that

$$\|Tx\| \le \max_{t\in[\omega_0,b]} \int_{\omega_0}^{b} |G(t,s)||f(s,x(s))| d_{q.\omega}s \le \frac{M}{\int_{\omega_0}^{b} \mathrm{G(s,}s)d_{q.\omega}s} \int_{\omega_0}^{b} \max_{t\in[\omega_0,b]} |G(t,s)| d_{q.\omega}s = M.$$

We see that $T : \mathrm{B}_M \to \mathrm{B}_M$. Consequently, we conclude by the Brouwer theorem that T has a fixed point $x_0 \in \mathrm{B}_M$ with $|x_0| \le M$.

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