International Journal of Latest Engineering and Management Research (IJLEMR)
ISSN: 2455-4847
www.ijlemr.com || Volume 02 - Issue 08 || August 2017 || PP. 51-60

A Case Study on Simple Harmonic Motion and Its Application

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Abstract: In this paper, we are going to study about simple harmonic motion and its applications. The simple harmonic motion of a spring-mass system generally exhibits a behavior strongly influenced by the geometric parameters of the spring. In this paper, we study the oscillatory behavior of a spring-mass system, considering the influence of varying the average spring diameter Φ on the elastic constant k, the angular frequency ω, the damping factor γ, and the dynamics of the oscillations. Simple harmonic motion and obtains expressions for the velocity, acceleration, amplitude, frequency and the position of a particle executing this motion. Its applications are clock, guitar, violin, bungee jumping, rubber bands,diving boards,eathquakes, or discussed with problems.
Keywords: Acceleration , Amplitude, Angular frequency, Velocity .

I. Introduction

In mechanics and physics, simple harmonic motion is a type of periodic motion or oscillation motion where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.

Simple harmonic motion can serve as a mathematical model for a variety of motions, such as the oscillation of a spring. In addition, other phenomena can be approximated by simple harmonic motion, including the motion of a simple pendulum as well as molecular vibration. Simple harmonic motion is typified by the motion of a mass on a spring when it is subject to the linear elastic restoring force given by Hooke's Law. The motion is sinusoidal in time and demonstrates a single resonant frequency. For simple harmonic motion to be an accurate model for a pendulum, the net force on the object at the end of the pendulum must be proportional to the displacement. This will be a good approximation when the angle of swing is small. Simple harmonic motion provides a basis for the characterization of more complicated motions through the techniques of Fourier analysis.

The motion of a particle moving along a straight line with an acceleration which is always towards a fixed point on the line and whose magnitude is proportional to the distance from the fixed point is called simple harmonic motion.

In the diagram, a simple harmonic oscillator, consisting of a weight attached to one end of a spring, is shown. The other end of the spring is connected to a rigid support such as a wall. If the system is left at rest at the equilibrium position then there is no net force acting on the mass. However, if the mass is displaced from the equilibrium position, the spring exerts a restoring elastic force that obeys Hooke's law.

Mathematically, the restoring force F is given by

$$F= -Kx$$

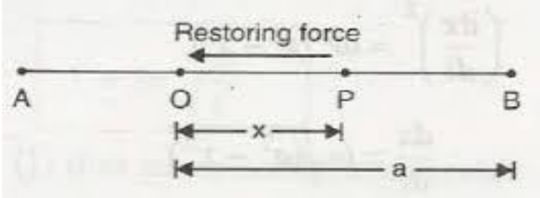
where F is the restoring elastic force exerted by the spring (in SI units: N), k is the spring constant ($N \cdot m^{-1}$), and x is the displacement from the equilibrium position (m).

II. Headings

The equation of motion of a particle executing simple harmonic motion, Geographical representation of simple harmonic motion, Composition of two simple harmonic motions of the same period along the same straight line, Composition of two simple harmonic motions of the same period in two perpendicular directions .

III. Indentations And Equations

The Equation of Motion of a Particle Executing Simple Harmonic Motion



Let O be the fixed point on the straight line AOB on which a particle is having simple harmonic motion. Take O as the origin and OA as the X axis. Let P be the position of the particle at time 't' such that OP=x. The magnitude of the acceleration at P=μx where μ is a positive constant. As this acceleration acts towards O, the acceleration at P in the positive direction of the x axis is -μx.

The magnitude of the acceleration at P is proportional to x that is the magnitude of the acceleration is $\mu x$where μ is a constant. As the acceleration is directed towards O.(i.e., in the direction of x decreasing).
Hence the equation of motion of P is,

$$\frac{d^2x}{dt^2} = -\mu x \rightarrow 1$$

Equation 1 is the fundamental differential equation representing a simple harmonic motion. We now proceed to solve it.
If V-velocity of the particle at time 't', 1 can be written as

$$v\frac{dv}{dx} = -\mu x$$

$$v.dv = -\mu x.dx \rightarrow 2$$

$$\frac{v^2}{2} = -\mu\frac{x^2}{2} + c \rightarrow 3$$

Initial value x=a, v=0,
Put in equation 3,

$$\frac{v^2}{2} = -\frac{\mu^2 x}{2} + c$$

$$0 = -\frac{\mu a^2}{2} + c$$

$$c = \frac{\mu a^2}{2}$$

$$\frac{v^2}{2} = -\frac{\mu x^2}{2} + \frac{\mu a^2}{2}$$

$$v^2 = -\mu x^2 + \mu a^2$$

$$v^2 = \mu(a^2 - x^2)$$

$$\therefore v = \pm\sqrt{\mu(a^2 - x^2)} \rightarrow 4$$

Equation 4 gives the velocity v corresponding to any displacement x.
Now as 't' increases, x decreases.
So, $\frac{dx}{dt}$ is negative.

So, take negative sign in 4

$$\frac{dx}{dt} = v = -\sqrt{\mu(a^2 - x^2)} \rightarrow 5$$

$$\frac{dx}{dt} = -\sqrt{\mu(a^2 - x^2)}$$

$$-\frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\mu}.dt$$

Integrating,

$$\cos^{-1}\frac{x}{a} = \sqrt{\mu}.t + A$$

Initially when t=0,x=a.

$$\cos^{-1} 1 = 0 + A$$

Hence

$$\cos^{-1}\frac{x}{a} = \sqrt{\mu}.t$$

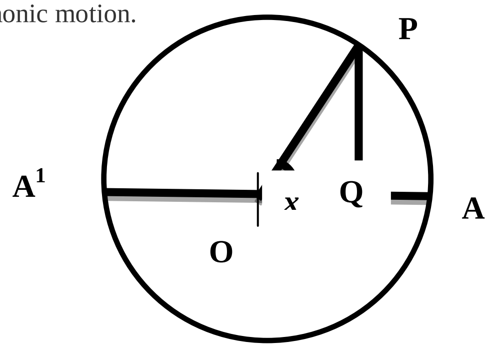
$$\frac{x}{a} = \cos\sqrt{\mu}.t$$

$$x = a\cos\sqrt{\mu}.t$$

This result gives the position of the particle at the end of 't' seconds, the time measured from the extreme position.

GEOMETRICAL REPRESENTATION OF SIMPLE HARMONIC MOTION:

Show that if a particle describes a circle with constant angular velocity, Then the foot of the perpendicular on a diameter moves with simple harmonic motion.



Let a particle move along the circumference of circle of radius a with uniform angular velocity ω. let AA^1 be a diameter of the circle. Let the position of the particle at time 't' be P. Then $\angle AOP = \omega t$.

Draw PQ $\perp^r$ to AA^1 and let OQ=x.

Then $x = \cos\omega t \rightarrow (1)$

As P moves on the circle Q moves o the diameter AA^1 that is Q oscillates between A and A^1 along AA^1. Therefore the motion of Q is simple harmonic motion.

From (1),

$$\frac{dx}{dt} = -a\omega \sin \omega t \rightarrow (2)$$

$$\frac{d^2x}{dt^2} = -a\omega^2 \cos \omega t$$

$$\frac{d^2x}{dt^2} = \omega^2 x \rightarrow (3)$$

Equation (2) gives the velocity of the particle Q and (3) gives the acceleration of Q at time T.
Also from (3) we note that the motion of Q is simple harmonic.
We know that the amplitude of the simple harmonic motion is a.

The periodic time of Q= $\frac{2\pi}{\omega}$

If a particle describes a circle with constant angular velocity then the foot of the perpendicular from it on any diameter executes simple harmonic motion.

Composition of Two Simple Harmonic Motions of the Same Period along the Same Straight Line

Let the two simple harmonic motions of the same period be given by

$$x = a\cos(\sqrt{\mu}t + \in_1)$$

$$x = b\cos(\sqrt{\mu}t + \in_2)$$

The composition of the two simple harmonic motions is

$$x = a\cos(\sqrt{\mu}t + \in_1) + b\cos(\sqrt{\mu}t + \in_2)$$

$$\cos(A+B) = \cos A.\cos B - \sin A..\sin B$$

$$x = a[\cos\sqrt{\mu}t.\cos \in_1 - \sin\sqrt{\mu}t.\sin \in_1] + b[\cos\sqrt{\mu}t.\cos \in_2 - \sin\sqrt{\mu}t.\sin \in_2]$$

$$x = a\cos\sqrt{\mu}t.\cos \in_1 - a\sin\sqrt{\mu}t.\sin \in_1 + b\cos\sqrt{\mu}t.\cos \in_2 - b\sin\sqrt{\mu}t.\sin \in_2$$

$$x = [a\cos \in_1 + b\cos \in_2]\cos\sqrt{\mu}t - [a\sin \in_1 + b\sin \in_2]\sin\sqrt{\mu}t$$

$$a\cos \in_1 + b\cos \in_2 = A\cos \in \rightarrow (I)$$

$$a\sin \in_1 + b\sin \in_2 = A\sin \in \rightarrow (II)$$

Then,

$$x = A\cos \in \cos\sqrt{\mu}t - A\sin \in \sin\sqrt{\mu}t$$

$$x = A(\cos \in \cos\sqrt{\mu}t - \sin \in \sin\sqrt{\mu}t)$$

$$x = A\cos(\sqrt{\mu}t + \in)$$

This equation shows that the composition of two simple harmonic motions is also a simple harmonic motion with the same period.

A is the amplitude and $\in$ is the epoch.
Dividing(II) by (I),

$$\frac{A\sin\in}{A\cos\in}=\frac{a\sin\in_1+b\sin\in_2}{a\cos\in_1+b\cos\in_2}$$

$$\tan\in=\frac{a\sin\in_1+b\sin\in_2}{a\cos\in_1+b\cos\in_2}\rightarrow(III)$$

Squaring and adding (I) and (II),

$$A^2\cos^2\in+A^2\sin^2\in=(a\cos\in_1+b\cos\in_2)^2+(a\sin\in_1+b\sin\in_2)^2$$

$$A^2(\cos^2\in+\sin^2\in)=a^2\cos^2\in_1+b^2\cos^2\in_2+2a\cos\in_1.b\cos\in_2+a^2\sin^2\in_1+b^2\sin^2\in_2+2a\sin\in_1.b\sin\in_2$$

$$A^2=a^2(\cos^2\in_1+\sin^2\in_1)+b^2(\cos^2\in_2+\sin^2\in_2)+2ab[\cos\in_1.\cos\in_2+\sin\in_1.\sin\in_2]$$

$$A^2=a^2+b^2+2ab\cos(\in_1-\in_2)$$

$$A=\sqrt{a^2+b^2+2ab\cos(\in_1-\in_2)}\rightarrow(IV)$$

(III) gives $\in$ and (IV) gives the amplitude A.

Composition of Two Simple Harmonic Motions of the Same Period in Two Perpendicular Directions

Let a particle execute simple harmonic motions along two perpendicular directions with the same period. Take two $\perp^r$ directions as x and y axes.
Let the displacement of the simple harmonic motions be given by

$$x=a_1\cos\sqrt{\mu}t\rightarrow(1)$$

$$y=a_2\cos(\sqrt{\mu}t+\in)\rightarrow(2)$$

The path of the particle is obtained by eliminating 't' from (1) and (2),

$$y=a_2\cos(\sqrt{\mu}t+\in)$$

$$y=a_2(\cos\sqrt{\mu}t.\cos\in-\sin\sqrt{\mu}t.\sin\in)$$

$$y=a_2\cos\sqrt{\mu}t.\cos\in-\sin\sqrt{\mu}t.\sin\in$$

$$y=a_2\cos t.\frac{x}{a_1}-a_2\sin t\sqrt{1-\frac{x^2}{a_1{}^2}}$$

$$y=a_2(\cos t.\frac{x}{a_1}-\sin t\sqrt{1-\frac{x^2}{a_1{}^2}})$$

$$\frac{y}{a_2}-\frac{x\cos t}{a_1}=-\sin\in\sqrt{1-\frac{x^2}{a_1{}^2}}$$

Squaring and simplifying we get

$$\left(\frac{y}{a_2}-\frac{x\cos\in}{a_1}\right)^2=\left(-\sin\in\sqrt{1-\frac{x^2}{a_1{}^2}}\right)^2$$

$$\frac{y^2}{a_2{}^2}-\frac{2xy\cos\in}{a_1a_2}+\frac{x^2\cos^2\in}{a_1{}^2}=\sin^2\in\left(1-\frac{x^2}{a_1{}^2}\right)$$

$$\frac{y^2}{a_2{}^2}-\frac{2xy\cos\in}{a_1a_2}+\frac{x^2\cos^2\in}{a_1{}^2}=\sin^2\in-\sin^2\in.\frac{x^2}{a_1{}^2}$$

$$\frac{y^2}{a_2{}^2}-\frac{2xy\cos\in}{a_1a_2}+\frac{x^2\cos^2\in}{a_1{}^2}+\frac{x^2\sin^2\in}{a_1{}^2}=\sin^2\in$$

$$\frac{y^2}{a_2{}^2}-\frac{2xy\cos\in}{a_1a_2}+\frac{x^2}{a_1{}^2}(\cos^2\in+\sin^2\in)=\sin^2\in$$

$$\frac{y^2}{a_2{}^2}-\frac{2xy\cos\in}{a_1\ a_2}+\frac{x^2}{a_1{}^2}=\sin^2\in\rightarrow(3)$$

This is a conic of the form

$$ax^2+2hxy+by^2=k$$

$$h^2-ab=\frac{\cos\in}{a_1{}^2a_2{}^2}-\frac{1}{a_1{}^2a_2{}^2}$$

$$h^2-ab=\sin^2\in<0$$

$\therefore$ Equation (3) is an ellipse.

NOTE 1:If $\in=0$ the path is a straight line.
NOTE 2:If $\in=\pi$ the path is a straight line.

NOTE 3:If $\in=\frac{\pi}{2}$,the path is an ellipse.

NOTE 4:If $\in=\frac{\pi}{2}$ and $a_1=a_2$,the path is circle.

IV. Problems

1) If v_1 and v_2 be the velocities of a particle moving in SHM at distances x_1 and x_2 from the centre show that the time of complete oscillation is $2\pi\sqrt{\frac{x_1{}^2-x_2{}^2}{v_1{}^2-v_2{}^2}}$.

Solution:
In a SHM,

$$v^2=\mu(a^2-x^2)$$

$$\therefore v_1{}^2=\mu(a^2-x_1{}^2)$$

$$\therefore v_2{}^2=\mu(a^2-x_2{}^2)$$

$$v_1{}^2-v_2{}^2=\mu a^2-\mu x_1{}^2\mu a^2+\mu x_2{}^2$$

$$v_1-v_2=\mu(x_2{}^2-x_1{}^2)$$

$$\frac{v_1{}^2-v_2{}^2}{x_2{}^2-x_1{}^2}=\mu$$

The period of oscillation= $\dfrac{2\pi}{\sqrt{\mu}}$

$$= \frac{2\pi}{\sqrt{\dfrac{v_1{}^2 - v_2{}^2}{x_2{}^2 - x_1{}^2}}}$$

$$= 2\pi\sqrt{\frac{x_2{}^2 - x_1{}^2}{v_1{}^2 - v_2{}^2}}$$

2) Show that in a SHM of amplitude a and period t, the velocity v at a distance x from the centre is given by the relation$v^2T^2 = 4\pi^2(a^2-x^2)$

Find the new amplitude if the velocity were doubled when the particle is at a distance $\dfrac{a}{2}$ from the centre, the period remaining the same.

Solution:

In a SHM,T= $\dfrac{2\pi}{\sqrt{\mu}} \rightarrow (1)$

$$v^2 = \mu(a^2 - x^2) \rightarrow (2)$$

Eliminating μ , from (1) and (2)

$$v^2 = \frac{4\pi^2}{T^2}(a^2 - x^2)$$

$$v^2T^2 = 4\pi^2(a^2 - x^2) \rightarrow (3)$$

Let $v=v_1$ when x= $\dfrac{a}{2}$

$$v_1{}^2T^2 = 4\pi^2(a^2 - \frac{a^2}{4})$$

$$v_1{}^2T^2 = 4\pi^2\left(\frac{4a^2 - a^2}{4}\right)$$

$$v_1{}^2T^2 = 3a^2\pi^2 \rightarrow (4)$$

A_1=amplitude when velocity at x= $\dfrac{a}{2}$ is doubled and the period remains the same.

i.e., $v = 2v_1, x = \dfrac{a}{2}, a = a_1$

Substituting in (3),

$$(2v_1)^2 T^2 = 4\pi^2 \left(a_1{}^2 - \frac{a^2}{4} \right)$$

$$4v_1{}^2 T^2 = 4\pi^2 \left(a_1{}^2 - \frac{a^2}{4} \right)$$

Substituting in (4),

$$3a^2\pi^2 = \pi^2 \left(a_1{}^2 - \frac{a^2}{4} \right)$$

$$3a^2 = a_1{}^2 - \frac{a^2}{4}$$

$$\frac{a^2}{4} + 3a^2 = a_1{}^2$$

$$\frac{a^2 + 12a^2}{4} = a_1{}^2$$

$$a_1{}^2 = \frac{13a^2}{4}$$

$$a_1 = \sqrt{\frac{13a^2}{4}}$$

$$a_1 = \frac{\sqrt{13}a}{2}$$

New amplitude= $\frac{a\sqrt{13}}{2}$

APPLICATIONS PROBLEMS:

1)The speed of waves in a particular guitar string is 425m/s. Determine the fundamental frequency of the string if its length is 76.5cm.

Solution:

V=425m/s
L=76.5cm=0.765m
F_1=?
Wavelength=2*length
=2*.765m
=1.53m
Speed=frequency x wave length
F =S/W

$F = \frac{425m/s}{1.53m}$
=278 Hz.

2)Determine the length of guitar string required to produce a fundamental frequency(first harmonic) of 256 Hz. The speed of waves in a particular guitar string is known to be 405m/s.

Solution:

V=405 m/s

F_1=256 Hz
S=f*w
W=s/f

W= $\frac{405m/s}{256Hz}$

W=1.58m
Length=1/2*wavelength
=0.791 m.

3)A guitar string with a length of 80.0cm is plucked. The speed of a wave in the string is 400m/s. calculate the frequency of the first, second and third harmonic.

Solution:
L=80.0cm=0.80 m
V=400m/s
λ =wavelength
λ =2*L
=2*(0.80)
=1.6 m
V=f* λ

F_1= $\frac{v}{\lambda}$

$= \frac{400m/s}{1.6m}$

$= 250Hz$

F_n=n*f_1
F_2=500Hz
F_3=750hz

4)A point on the string of a violin moves up and down in simple harmonic motion with an amplitude of 1.24mm and a frequency of 875hz.
a)what is the maximum speed of that point in SI units?
b)what is the maximum acceleration of the point in SI units?

Solution:
Amplitude (a)=1.24mm
Frequency=875hz
Maximum speed=2πfA
=2 x 3.14 x 875 x 1.24
=6813.8mm
=6.82 m/s
Maximum acceleration= $(2\pi f)^2$A
=$(2 \times 3.14 \times 875)^2 \times 1.24$
=37441831
=3.75x10^4 m/s.

5)Most grandfather clock have pendulums with adjustable lengths one such clock loses 10 min per day when the length of its pendulum is 30in with what length pendulum will this clock keep perfect time?

Solution:
A relationship between the period and length of the pendulum must be developed for the two situations. According, circular frequency of a pendulum is given by $\omega^2 = \frac{g}{L} = \frac{GM}{R^2 L}$. therefore, the period P= $\frac{2\pi}{\omega} = 2\pi R\sqrt{\frac{L}{GM}}$

Dividing this equation for period by another period with length gives the necessary relationship for this problem:

$$\frac{P_1}{P_2} = \frac{\sqrt{L_1}}{\sqrt{L_2}}$$

This given information is as follows:

$L_1=30$

Assuming the pendulum executes n cycles.
The pendulum executes n cycles per day
$P_2=1440/m$
Since the pendulum takes 10min longer to carry out the same number of cycles, the period when the pendulum loses 10min is,
$P_1=1450/mins$

$$L_2=\frac{L_1 P_2{}^2}{P_1{}^2}$$

$$=\frac{30(1440)^2}{(1450)^2}$$

$= 29.6$ in

Conclusion

In this case study simple harmonic motion and its applications. Different applications problems are solved analytically with exact equation of simple harmonic motion. We can calculate the periodic time value of oscillating a object from origin by this methods.

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