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Semi-Strong Setsin a Graph

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Abstract: Let $G = (V, E)$ be a simple, finite undirected graph. A subset S of $V(G)$ is called a semi-strong stable set if $|\ N(v)\ \cap S|\ \leq 1$for every v in $V(G)$[5]. The hereditary property of a semi-strong stable set is used to define two parameters.A subset S of $V(G)$ is called a maximal semi-strong set of$G$ if S is semi-strong and no proper super set of S is semi-strong. The maximum cardinality of a maximal semi-strong set of G is called semi-strong number of G and is denoted by $ss(G)$. The minimum cardinality of a maximal semi-strong set of G is called the lower semi-strong number of $G$and is denoted by $lss(G)$. In this paper we study the bounds of the above two parameters of standard graphsand the related characterization.

Keywords:strong stable set, semi strong set, semi strong number
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Introduction

Claude Berge[1] introduced the concept of strong stable set in a graph. Let $G = (V, E)$ be a simple, finite undirected graph. A subset S of $V(G)$ is called a strong stable set of G if $|N[v]\ \cap S|\ \leq 1$ for every$v$ in$V(G)$. It can be easily seen that such a set is independent and the distance between any two vertices of $S$is greater than or equal to three. That is, a strong stable set is a 2-packing. Generalizing this concept, E.Sampathkumar and L.Pushpa Latha [5]introduced semi-strong sets. A subset S of $V(G)$ is called semi-strong stable if$|N(v)\ \cap S| \leq 1$for every$v$ in $V(G)$. A strongstable set is semi-strong stable but the converse is not true. For example,in a cycle of order 5, C_5, any two consecutive vertices is a semi-strong stable set. If S is a semi-strong stable set, then any component of S is either K_1 or K_2 and the distance between any two points of S is not equal to two. E.Sampathkumar and L.Pushpa Latha discusseda partitionof V(G) into semi-strong stable sets. Such a partition is called semi-strong stable coloring. Semi-strong chromatic number of a graph has been defined and several results were derived. In this paper, the property of being aSemi-strong stable set is observed to be hereditary and this property of semi-strong stable set is used to define two parameters namely maximum cardinalityof a maximal semi-strong stable set ($ss(G)$) and minimum cardinalityof a maximalsemi-strongstable set($lss(G)$).

Observation: The property of being a semi-strong set is hereditary.

Definition 1: A subset S of $V(G)$ is called a maximal semi-strong set of G if S is semi-strong and no proper super set of S is semi-strong. The maximum cardinality of a maximalsemi-strong set of G is called the semi-strong number of G and is denoted by$ss(G)$. The minimum cardinality of a maximal semi-strong setof G is called the lower semi-strong number of G and is denoted by $lss(G)$.
Example: Let $G = C_9$. Let $V(G) = \{u_1, u_2, \ldots, u_9\}$. $S = \{u_1, u_2, u_5, u_6\}$ is a maximum semi-strong set of G. Therefore $ss(G)$ = 4. $\{u_1, u_4, u_7\}$ is a maximal semi-strong set of G which is not maximum.

$\boldsymbol{ss(G)}$ and $\boldsymbol{lss(G)}$ for someWell-known Graphs:

1.$ss(K_n) = \begin{cases} 2 & if\ n = 2 \\ 1 & if\ n \geq 3 \end{cases}$
2. $ss(K_{m,n})$ = 2 where m,n ≥ 1
3. $ss(K_{1,n})$ = 2 where n ≥ 1

4. $ss(P_n)=\begin{cases} \frac{n}{2}+1 \; if \; n \equiv 2(mod\; 4) \\ \left\lceil \frac{n}{2} \right\rceil \quad otherwise \end{cases}$
5. $ss(W_n) = 1$
6. $ss(C_n)=\begin{cases} \frac{n}{2}-1 \; if \; n \equiv 2(mod\; 4) \\ \left\lfloor \frac{n}{2} \right\rfloor \quad otherwise \end{cases}$
7. ss(P) = 2, where P is the Petersen Graph.
8. $ss(K_m(a_1, a_2, \dots, a_m)) = m$
9. $ss(K_{a_1,a_2,\dots,a_n}) = 1$, if $n \geq 3$.

1. $lss(K_n) = \begin{cases} 2 \; if \; n = 2 \\ 1 \; if \; n \geq 3 \end{cases}$
2. $lss(K_{m,n}) = 2$ where $m, n \geq 1$
3. $lss(K_{1,n}) = 2$ where $n \geq 1$
4. $lss(P_n)=\begin{cases} \frac{n}{2}+1 \; if \; n \equiv 2(mod\; 4) \\ \left\lfloor \frac{n}{2} \right\rfloor \quad otherwise \end{cases}$
5. $lss(W_n) = 1$
8. $lss(K_m(a_1, a_2, \dots, a_m)) = 2$
9. $lss(K_{a_1,a_2,\dots,a_n}) = 1$, if $n \geq 3$

Observation 1: $1 \leq lss(G) \leq ss(G) \leq n.$

Observation 2: If G has a full degree vertex and $|V(G)| \geq 3$, then $ss(G) \leq 2$.
Proof: Let u be a full degree vertex of G. Let $|V(G)| \geq 3$. Let $V(G) = \{u, v_2, v_3, \dots, v_n\}$. LetS be a maximum semi-strong setof G. For any i, j, and $i \neq j$, $(2 \leq i \leq n)$, v_i and v_j together cannot belongs to S. Suppose $v_i \in S$.Suppose v_j is notadjacent with $v_i (i \neq j)$. In this case, $ss(G) = 2$. If for every i, there exist some j, such that v_j is adjacent with v_i, then v_i, u together cannot belong to S. Therefore, in this case $ss(G) = 1$. Therefore $ss(G) \leq 2$.

Corollary 1: If G has a full degree vertex and $|V(G)| \geq 3$, then $ss(G) = 2$ if and only if G is a star.

Corollary 2: If G has afulldegree vertex u and $|V(G)| \geq 3$, then $ss(G) = 1$ if and only if $< V(G) - \{u\} >$ has no isolates.

Theorem 1: Let $|V(G)| \geq 3$. Then $ss(G) = 1$ if and only if any two vertices of G have a common vertex in G.
Proof: By hypothesis, no two vertices of G form a semi-strong set.Therefore $ss(G) = 1$. The converse isobvious.

Definition 2: $N(G)$, called the neighbourhood graph G has the same vertex set as G and two vertices in $N(G)$ are adjacent if and only if they have a common neighbour.

Theorem 2: Let G be a graph with atleast three vertices. Then $ss(G) = 1$ if and only if either G has a full degree vertex say u such that $< V(G) - \{u\} >$ has no isolates or G is a multipartite graphwith atleast three partite sets such that $N(G) = K_n$.
Proof: Let $ss(G) = 1$. Suppose G has a full degree vertex say u.Since $|V(G)| \geq 3$, by Corollary2, $< V(G) - \{u\} >$ has no isolates. Suppose G has no full degree vertex. Clearly $diam(G) = 2$ and everyedge is on a triangle.Let $u_1 \in V(G)$. Let it be $V_1 = \{v_1, v_2, \dots, v_{k_1}\}$. If $V_1 = V(G)$, then $G = K_{k_1}$ and hence $ss(G) = k_1 \geq 2$, (since u_1 is not a full degree vertex), a contradiction. Therefore $V_1 \subset V(G)$.Let V_2 be a maximal independent set containing v_1. Let $V_2 = \{v_1, v_2, \dots, v_{k_2}\}$. If $V_1 \cup V_2 = V(G)$, then G is bipartite andhence $ss(G) \geq 2$, a contradiction. Therefore there exist $w_1 \in V(G) - (V_1 \cup V_2)$.Let V_3 be a maximal independent set containing w_1. Let $V_3 = \{w_1, w_2, \dots, w_{k_3}\}$. Suppose $V(G)=V_1 \cup V_2 \cup V_3$. Since $ss(G) = 1$, $N(G) = K_n$.

If $V(G) \supset V_1 \cup V_2 \cup V_3$, proceeding as before we arrive at a multipartite graph withatleast threevertices such that $N(G) = K_n$.The converse is obvious.

Theorem 3: $ss(G) = n$ if and only if every component of G is either K_1 or K_2.
Proof: Let $ss(G) = n$. Then $V = \{u_1, u_2, \dots, u_n\}$ is a ss-set of G. Therefore G is P_3-free and K_3-free. Thedistance betweenany twovertices cannot be greater than or equal to two. Therefore either u_i and u_j are adjacentor u_i and u_j independent. If u_i and u_j are adjacent, then there exist no vertex u_k which is adjacent with either u_i or$u_j$or both. Therefore $< u_i , u_j >$ is a component of G. If u_i and u_j are independent, then there exists a vertex u_k adjacent with u_i and u_j. If u_k is adjacent with u_i, then $< u_i , u_k >$ is a component of G. If u_k is adjacent with u_j, then $< u_j , u_k >$ is a component of G. Since $ss(G) = n$ any u_i can be isolate of G or u_i forms a K_2-component with some vertex of G.The converse is obvious.

Remark 1: If G is connected, then $ss(G) = n$ if and only if $n = 1$ or 2.

Theorem 4: Let G be any graph. Then $ss(G) = 2$ if and only if there exist two vertices u_1, u_2 (independent or adjacent) such that any u_i,($3 \leq i \leq n$), is adjacent with exactly one of u_1,u_2 and in case u_1,u_2 are independent, either $< \{u_3, u_4, \dots, u_n\} >$ has no isolates provided atleast two vertices from $u_3, u_4, \dots, u_n$are adjacent with u_1, so also with u_2, or u_3 is adjacent with every u_i, ($4 \leq i \leq n$) where u_3 is the only vertex from $u_3, u_4, \dots, u_n$which is adjacent with u_1 and $u_4, u_5, \dots, u_n$are adjacent with u_2.
Proof: Let S= { u_1,u_2 } be a ss-set of G. Then for any u_i,($3 \leq i \leq n$), u_i is adjacent with exactlyone of u_1,u_2 .Let $u_3, u_4, \dots, u_r$ be adjacent with u_1 and $u_{r+1}, u_{r+2}, \dots, u_n$be adjacent with u_2.
Subcase 1: $r \geq 2$ and $n - r \geq 2$.

Then any two vertices adjacent with u_1 do not form a semi-strong set.So also any vertex adjacent with u_2. Also any semi-strong set of G from $\{u_3, u_4, \dots, u_n\}$ cannot contain more than two vertices, provided$< u_3, u_4, \dots, u_n >$ has no isolates, in this case u_1 and u_2 are independent.
Subcase 2:$r = 1$ or $r = n - 3$.

Suppose ss(G) = 2. Let S= { u_1,u_2 } be a ss-set of G. Then for any u_i,($3 \leq i \leq n$), u_i is adjacent withexactly one of u_1,u_2 . Let $u_3, u_4, \dots, u_r$ be adjacent with u_1 and $u_{r+1}, u_{r+2}, \dots, u_n$be adjacent with u_2.
Subcase 1: $r \geq 2$ and $n - r \geq 2$.

Since {u_1,u_2} is a a ss-setof G, and since u_1,u_2 are adjacent for any subgraphinduced by $u_3, u_4, \dots, u_n$, {u_1,u_2} is a ss-set of G and ss(G) = 2.
Subcase 2:$r = 1$ or $r = n - 3$.

In this case also, ss(G) = 2 for any subgraph induced by $u_3, u_4, \dots, u_n$.The converse is obvious.

Theorem 5: Let G be a graph. Then $ss(G) = n - 1$ if and only if there exists exactly one P_3 component andother components areeither K_1 or K_2.
Proof: Let $ss(G) = n - 1$. Let $V(G) = \{u_1, u_2, \dots, u_n\}$. Let S be a ss-set of G. Let $S = \{u_1, u_2, \dots, u_{n-1}\}$. Any component of S is either K_1 or K_2.$|N(u_n) \cap S| \leq 1$. If u_n is not adjacent with any vertex of S, then $S \cup \{u_n\}$ is a ss-set of G, a contradiction. If u_n is adjacent with exactly one K_1 component of S, then again $S \cup \{u_n\}$ is a ss-set of G, a contradiction. If u_n is adjacent with exactly one K_2 component of S. That is, $S \cup \{u_n\}$ contains exactly one P_3. That is, every component of G is either K_1 or K_2 or P_3, the P_3 component beingunique.The converse is obvious.

References:

[1]. C. Berge,Graphs and Hyper graphs, North Holland, Amsterdam, 1973.
[2]. F.Harary, Graph Theory, Addison Wesley, 1969.
[3]. T.W.Haynes,S.T.Hedetniemi and P.J. Slater, Fundamentals of domination in graphs, Marcel Dekker, New York, 1998.
[4]. G. Jothilakshmi, A. P. Pushpalatha, S. Suganthi and V. Swaminathan, (k,r) - Semi Strong Chromatic Number of a Graph, International Journal of Computer Applications, Vol. 21, No. 2, 2011, Pages 7-10.
[5]. E. Sampathkumar and L. Pushpa Latha, Semi-Strong Chromatic Number of a Graph, Indian Journal of Pure and Applied Mathematics, 26 (1): 35-40, 1995.