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Lyapunov-Type Inequalities for the Quasilinear *q*-Symmetric Difference Systems

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Abstract: Using the H□lder inequality, we establish several Lyapunov-type inequalities for quasilinear *q*-symmetric equation and *q*-symmetric difference systems.

Keywords: Lyapunov-type inequality, *q*-symmetric difference equation, *q*-symmetric difference systems, H□lder inequality.

I. Introduction

The Lyapunov inequality and many of its generalizations have proved to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications for the theories of differential and difference equation. The well-known inequality of Lyapunov [1] states that a necessary condition for the boundary value problem $y''+q(t)y=0, y(a)=y(b)=0,$ to have nontrivial solutions is that $\int_a^b |q(t)|dt > \frac{4}{b-a}$. There are several different proofs of this inequality since the original one by Lyapunov [1]. In the last few years independent works appeared generalizing Lyapunov's inequality for the *p*- Laplacian, by using H□lder, Jenson or Cauchy-Schwarz inequalities. For the case of a single equation, see for example [2-9]. For system there are a few results, see for [10-11]. On the other hand, the *q*-calculus and *q*-symmetric calculus have proven to be useful in several fields, in particular in quantum mechanics [12]. However, there are very a few results for *q*-symmetric difference system [13-14]. In the paper, we consider boundary value problem of the following quasilinear *q*-symmetric difference equation.

$$\begin{cases} -D_q\left(r(t)\left|\overset{\square}{D}_q u(t)\right|^{p-2} (\overset{\square}{D}_q u(t))\right) = f(t)|u(t)|^{p-2}u(t), t\in(0,1), 0<q<1 \\ u(0)=u(q)=0, u(t)\neq 0, t\in(0,q). \end{cases}, \qquad (1)$$

And

$$\begin{cases} -\overset{\square}{D}_q(r_1(t)|u(t)|^{p_1-2}(\overset{\square}{D}_q u(t))) = f_1(t)|u(t)|^{\alpha_1-2}u(t)|v(t)|^{\alpha_2}, t\in(0,1), 0<q<1, \\ -\overset{\square}{D}_q(r_2(t))\overset{\square}{D}_q|v(t)|^{p_2-2}(\overset{\square}{D}_q v(t)) = f_2(t)|v(t)|^{\beta_2-2}v(t)|u(t)|^{\beta_1}, t\in(0,1), 0<q<1, \\ u(0)=u(q)=v(0)=v(q)=0, u(t)\neq 0, v(t)\neq 0, t\in(0,q). \end{cases} \qquad (2)$$

For the save of convenience, we give the following hypothesis (H_1) and (H_2) for (1) and hypothesis (H_3) for (2):

(H_1) $r(t)$ and $f(t)$ are real-valued functions and $r(t)>0$ for all $t\in R$, $1<p, p_1<\infty$, satisfy

$$\frac{1}{p_1}+\frac{1}{p}=1.$$

(H_2) $1<p_1,p_2<\infty,\alpha_1,\alpha_2,\beta_1,\beta_2>0$ satisfy $\frac{\alpha_1}{p_1}+\frac{\alpha_2}{p_2}=1$ and $\frac{\beta_1}{p_1}+\frac{\beta_2}{p_2}=1$.

We recall some concepts for *q*-symmetric difference operator.

Throughout this paper, we assume $q\in(0,1)$.

The *q*-symmetric derivatives $\tilde{D}_q f$ of a function f is given by

$$\left(\tilde{D}_q f\right)(x)=\frac{f(qx)-f(q^{-1}x)}{(q-q^{-1})x},\left(\tilde{D}_q f\right)(0)=f'(0),$$ provided $f'(0)$ exists.

The *q*-symmetric integrals is defined by

$$\int_0^a f(x)\tilde{d}_q x=(1-q^2)a\sum_{n=0}^{\infty}q^{2n}f(aq^{2n+1}),$$

$$\int_a^b f(x)\tilde{d}_q x=\int_0^b f(x)\tilde{d}_q x-\int_0^a f(x)\tilde{d}_q x.$$

The *q*-symmetric analogue of Leibnitz rule is given by

$$\tilde{D}_q f(qx)g(x)=g(qx)D_q f(qx)+f(x)D_q g(x). \quad (3)$$

II. Lyapunov-Type inequalies for *q*-Symmetric Difference Equation (1)

In the section, we establish Lyapunov-type inequalies for *q*-symmetric difference equation (1).
Denote

$$\xi(t)=(\int_0^t r^{1-p_1}(s)\tilde{d}_q s)^{\frac{1}{p_1-1}}$$

$$\eta(t)=(\int_t^1 r^{1-p_1}(s)\tilde{d}_q s)^{\frac{1}{p_1-1}}$$

Theorem 2.1. Suppose that hypothesis (H_3) holds. If the boundary value problem (1) has a solution. Then one has the following inequality

$$\int_0^1\frac{\xi(t)\eta(t)}{\xi(t)+\eta(t)}f^+(t)\tilde{d}_q t\geq q^{-1}$$

where $f^+(t)=\max\{f(t),0\}$.

Proof. By (1) and (3), we have

$$\int_0^1 f_1(t)\left|u(t)\right|^p \tilde{d}_q t$$
$$=-\int_0^1 \tilde{D}_q\left(r(t)\left|\tilde{D}_q u(t)\right|^{p-2}\left(\tilde{D}_q u(t)\right)\right)u(t)\tilde{d}_q t$$
$$=\int_0^1 r(qt)\left|\tilde{D}_q(qt)\right|^{p-2}\tilde{D}_q u(qt)\tilde{D}_q u(qt)\tilde{d}_q t-\int_0^1 \tilde{D}_q(u(qt)\left((r(t)\left|\tilde{D}_q u(t)\right|^{p-2}\tilde{D}_q u(t))\right)\tilde{d}_q t$$
$$=\int_0^1 r(qt)\left|\tilde{D}_q(qt)\right|^p \tilde{d}_q t$$
$$=q^{-1}\int_0^q r(s)\left|\tilde{D}_q(u(s))\right|^p \tilde{d}_q s.$$

By the boundary condition of (1), we can get

$$\left|u(qt)\right|^p=\left|\int_0^{qt}\tilde{D}_q u(s)\tilde{d}_q s\right|^p \le (\int_0^{qt}\left|\tilde{D}_q u(s)\right|\tilde{d}_q s)^p$$
$$=(\int_0^{qt} r^{-\frac{1}{p}}(s) r^{\frac{1}{p}}(s)\left|\tilde{D}_q u(s)\right|\tilde{d}_q s)^p$$
$$\le (\int_0^{qt} r^{1-p}(s)\tilde{d}_q s)^{\frac{1}{p_1-1}}\int_0^{qt} r(s)\left|\tilde{D}_q u(s)\right|^p \tilde{d}_q s$$
$$=\xi(qt)\int_0^{qt} r(s)\left|\tilde{D}_q u(s)\right|^p \tilde{d}_q s$$

and

$$\left|u(qt)\right|^p=\left|-\int_{qt}^q \tilde{D}_q u(s)\tilde{d}_q s\right|^p \le \left[\int_{qt}^q\left|\tilde{D}_q u(s)\right|\tilde{d}_q s\right]^p$$
$$=\left(\int_{qt}^q r^{1-p}(s)\tilde{d}_q s\right)^{\frac{1}{p-1}}\left(\int_{qt}^q r(s)\left|\tilde{D}_q u(s)\right|^p \tilde{d}_q s\right)\le\left(\int_{qt}^1 r^{1-p_1}(s)\tilde{d}_q s\right)^{\frac{1}{p-1}}\left(\int_{qt}^q r(s)\left|D_q u(s)\right|^p \tilde{d}_q s\right)$$
$$=\eta(qt)\int_{qt}^q r(s)\left|\tilde{D}_q u(s)\right|^p \tilde{d}_q s.$$

Thus

$$\left|u(qt)\right|^p \le \frac{\xi(qt)\eta(qt)}{\xi(qt)+\eta(qt)}\int_0^q r(s)\left|\tilde{D}_q u(s)\right|^p \tilde{d}_q s.$$

Therefore

$$\left|u(t)\right|^p \le \frac{\xi(qt)\eta(qt)}{\xi(qt)+\eta(qt)}\int_0^q r(s)\left|\tilde{D}_q u(s)\right|^p \tilde{d}_q s$$

By (4), we have

$$\int_0^1 f^+(t)\left|u(t)\right|^p \tilde{d}_q t \le \int_0^1 \frac{\xi(qt)\eta(qt)}{\xi(qt)+\eta(qt)} f^+(t)\tilde{d}_q t\cdot\int_0^q r(s)\left|\tilde{D}_q u(s)\right|^p \tilde{d}_q s$$

$$= q\int_0^1 \frac{\xi(qt)\eta(qt)}{\xi(qt)+\eta(qt)} f^+(t)\tilde{d}_q t \cdot \int_0^1 f_1(t)|u(t)|^p \tilde{d}_q t.$$

$$\leq q\int_0^1 \frac{\xi(qt)\eta(qt)}{\xi(qt)+\eta(qt)} f^+(t)\tilde{d}_q t \cdot \int_0^1 f^{-1}(t)|u(t)|^p \tilde{d}_q t . \quad (5)$$

Next, we prove that $\int_0^1 f^+(t)|u(t)|^p \tilde{d}_q t > 0.$ (6)

If (6) is not true, then $\int_0^1 f^+(t)|u(t)|^p \tilde{d}_q t = 0$ (7)

From (4) and (7), we have

$$0 \leq \overset{-1}{q}\int_0^q r(s)\left|\tilde{D}_q u(s)\right|^p \tilde{d}_q s = \int_0^1 f(t)|u(t)|^p \tilde{d}_q t \leq \int_0^1 f^+(t)|u(t)|^p \tilde{d}_q t = 0.$$

It follows $\tilde{D}_q u\left(q^{2n+1}t\right) \equiv 0, n = 0,1,...,$, we obtain that $u(t) \equiv 0$, for $t \in (0,q)$ which contradicts the condition (2). Therefore, from (5), we may see that theorem 2.1 holds.

Note that $(\frac{\xi+\eta}{2})^2 \geq \xi\eta$, one has following corollary 2.1.

Corollary 2.1. Suppose that hypothesis (H_1) is satisfied. If (1) has a solution $u(t)$. Then one has the following inequality.

$$\int_0^1 (\xi(t)\eta(t))^{\frac{1}{2}} f^+(t)\tilde{d}_q t \geq 2q^{-1}$$

Ⅲ Lyapunov-type Inequalities for *q*-symmotric difference system(2)

Denote

$$\xi_i(t) = (\int_0^t r_i^{1-p_i}(s)\tilde{d}_g s)^{\frac{1}{p_i-1}}, i = 1,2 \quad (8)$$

$$\eta_i(t) = (\int_t^1 r_i^{1-p_i}(s)\tilde{d}_g s)^{\frac{1}{p_i-1}}, i = 1,2 \quad (9)$$

Theorem 3.1. Suppose that hypothesis (H_2) is satisfied. If system (2) has a solution $(u(t),v(t))$. Then one has the following inequality:

$$M_{11}^{\alpha_1\beta_1/p_1^2} M_{12}^{\beta_1\alpha_2/p_1p_2} M_{21}^{\beta_1\alpha_2/p_1p_2} M_{22}^{\alpha_2\beta_2/p_2^2} \geq q^{-(\beta_1/p_1+\alpha_2/p_2)}$$

where $M_{ij} = \int_0^1 \frac{\xi_i(t)\eta_i(t)}{\xi_i(t)+\eta_i(t)} f_j^+(t)\tilde{d}_q t \quad i, j = 1,2$

where $f_j^+(t) = \max\{f_i(t), 0\}$, for $i = 1, 2$

Proof. Similar to (4), we have

$$\int_0^q r_1(s)\,|\tilde{D}_q u(s)|^{p_1}\,\tilde{d}_q s = q\int_0^1 f_1(s)\,|u(s)|^{\alpha_1}|v(s)|^{\alpha_2}\,\tilde{d}_q s \tag{10}$$

$$\int_0^q r_2(s)\,|\tilde{D}_q v(s)|^{p_2}\,\tilde{d}_q s = q\int_0^1 f_2(s)\,|v(s)|^{\beta_2}|u(s)|^{\beta_1}\,\tilde{d}_q s \tag{11}$$

$$|u(t)|^{P_1} \le \frac{\xi_1(t)\eta_1(t)}{\xi_1(t)+\eta_1(t)}\int_0^q r_1(s)\,|\tilde{D}_q u(s)|^{P_1}\,\tilde{d}_q s \tag{12}$$

$$\int_0^1 \int_1^+(t)\,|u(t)|^{P_1}\,\tilde{d}_q t \le M_{11}\int_0^q r_1(s)\,|\tilde{D}_q u(s)|^{p_1}\,\tilde{d}_q s$$

$$= qM_{11}\int_0^1 f_1(s)\,|u(s)|^{\alpha_1}|v(s)|^{\alpha_2}\,\tilde{d}_q s$$

$$\le qM_{11}\int_0^1 f_1^+(s)\,|u(s)|^{\alpha_1}|v(s)|^{\alpha_2}\,\tilde{d}_q s$$

$$\le qM_{11}\Big(\int_0^1 f_1(s)|u(s)|^{p_1}\,\tilde{d}_q s\Big)^{\frac{\alpha_1}{p_1}}\Big(\int_0^1 f_1^+(s)|v(s)|^{p_2}\,\tilde{d}_q s\Big)^{\frac{\alpha_2}{p_2}} \tag{13}$$

$$\int_0^1 f_2^+(t)|u(t)|^{p_1}\,\tilde{d}_q t \le qM_{12}\Big(\int_0^1 f_1^+(s)|u(s)|^{p_1}\,\tilde{d}_q s\Big)^{\frac{\alpha_1}{p_1}}\Big(\int_0^1 f_1^+(s)|v(s)|^{p_2}\,\tilde{d}_q t\Big)^{\frac{\alpha_2}{p_2}} \tag{14}$$

Similar to the proof of (12), from (10) (11), we have

$$|v(s)|^{p_2} \le \frac{\xi_2(t)\eta_2(t)}{\xi_2(t)+\eta_2(t)}\int_0^q r_2(s)|D_q v(s)|^{p_2}\,d_q s$$

It follows from above form and the H□lder inequality that

$$\int_0^1 f_1^+(t)|v(t)|^{p_2}\,\tilde{d}_q t \le \int_0^1 \frac{\xi_2(t)\eta_2(t)}{\xi_2(t)+\eta_2(t)} f_1^+(t)\,\tilde{d}_q t\int_0^q r_2(s)|D_q v(s)|^{p_2}\,d_q s$$

$$\le qM_{12}\Big(\int_0^1 f_2^+(t)|u(t)|^{\beta_1}|v(t)|^{\beta_2}\,\tilde{d}_q t\Big) \tag{15 }$$

$$\le qM_{12}\Big(\int_0^1 f_2^+(t)|u(t)|^{p_1}\,\tilde{d}_q s\Big)^{\frac{\beta_1}{p_1}}\Big(\int_0^1 f_2^+(t)|v(t)|^{p_2}\,\tilde{d}_q t\Big)^{\frac{\beta_2}{p_2}}$$

$$\int_0^1 f_2^+(t)|v(t)|^{p_2}\,\tilde{d}_q t \le qM_{12}\Big(\int_0^1 f_2^+(t)|u(t)|^{p_1}\,\tilde{d}_q t\Big)^{\frac{\beta_1}{p_1}}\Big(\int_0^1 f_2^+(t)|v(t)|^{p_2}\,\tilde{d}_q t\Big)^{\frac{\beta_2}{p_2}} \tag{16}$$

Similar to (5), we can get

$\int_0^1 f_1^+(t)|u(t)|^{p_1}\,\tilde{d}_q t > 0$, $\int_0^1 f_2^+(t)|u(t)|^{p_1}\,\tilde{d}_q t > 0$,

$\int_0^1 f_1^+(t)|v(t)|^{p_1}\,\tilde{d}_q t > 0$, $\int_0^1 f_2^+(t)|v(t)|^{p_2}\,\tilde{d}_q t > 0$.

From (14)-(17), we have

$$M_{11}^{\frac{\alpha_1\beta_1}{p_1^2}} M_{12}^{\frac{\beta_1\alpha_2}{p_1p_2}} M_{21}^{\frac{\beta_1\alpha_2}{p_1p_2}} M_{22}^{\frac{\alpha_2\beta_2}{p_2^2}} \geq q^{-(\frac{\beta_1}{p_1}+\frac{\alpha_2}{p_2})}$$

Corollary3.1. Suppose that hypothesis (H_2) are satisfied. If (2) has a solution $(u(t), v(t))$. Then

$$(\int_0^1 (\xi_1(t)\eta_1(t))^{\frac{1}{2}} f_1^+ d_q t)^{\frac{\beta_1\alpha_2}{p_1^2}} (\int_0^1 (\xi_2(t)\eta_2(t))^{\frac{1}{2}} d_q(t))^{\frac{\beta_1\alpha_2}{p_1p_2}}$$

$$\times(\int_0^1 (\xi_1(t)\eta_1(t))^{\frac{1}{2}} f_1^+ d_q t)^{\frac{\beta_1\alpha_2}{p_1p_2}} (\int_0^1 (\xi_2(t)\eta_2(t))^{\frac{1}{2}} d_q(t))^{\frac{\beta_1\alpha_2}{p_2^2}}$$

Next, we consider the quasilinear q-symmetric system involving the ($P_1, P_2, \cdots, P_m$)-Laplacian:

$$\begin{cases} -\tilde{D}_q(r_1(t)\left|D_q u_1(t)\right|^{p_1-2}(D_q u_1(t))) = f_1(t)\left|u_1(t)\right|^{\alpha_1-2}\left|u_2(t)\right|^{\alpha_2}\cdots\left|u_m(t)\right|^{\alpha_m} u_1(t) \\ -\tilde{D}_q(r_2(t)\left|D_q u_2(t)\right|^{p_2-2}(D_q u_2(t))) = f_2(t)\left|u_1(t)\right|^{\alpha_1}\left|u_2(t)\right|^{\alpha_2-2}\cdots\left|u_m(t)\right|^{\alpha_m} u_2(t) \\ \vdots \\ -\tilde{D}_q(r_m(t)\left|D_q u_m(t)\right|^{p_m-2}(D_q u_m(t))) = f_m(t)\left|u_1(t)\right|^{\alpha_1}\left|u_2(t)\right|^{\alpha_2}\cdots\left|u_m(t)\right|^{\alpha_m-2} u_m(t) \end{cases} \quad (18)$$

with boundary value conditions:

$$u_i(0) = u_i(q) = 0, u_i(t) \neq 0, t \in (0, q), i = 1, 2, \cdots, \mathrm{m}. \quad (19)$$

We give the following hypothesis (H_3).

(H_3) $r_i(t)$ and $f_i(t)$ are real-valued functions and $r_i(t) \geq 0$ for $i = 1, 2, ..., m$.

Furthermore, $1 < p_i < \infty$ and $\alpha_i > 0$ satisfy $\sum_{i=1}^{m}(\frac{\alpha_i}{p_i}) = 1$.

Denote $$\xi_i(t) = (\int_0^t r_i^{1-p_i}(\mathrm{s})\, d_q s)^{\frac{1}{p_i-1}}$$

$$\eta_i(t) = (\int_t^1 r_i^{1-p_i}(\mathrm{s})\, d_q s)^{\frac{1}{p_i-1}} \quad (20)$$

Theorem 3.2. Suppose that hypothesis (H_3) is satisfied. If system (18) has a solution $(u_1(t), u_2(t), ...u_m(t))$ satisfying the boundary condition (19), then one has the following inequality:

$$\coprod_{i=1}^{m}\prod_{j=1}^{m}(\int_0^1 \frac{\xi_i(\tau)\eta_i(\tau)}{\xi_i(\tau)+\eta_i(\tau)} f_j^+(\tau)\, d_q\tau)^{\frac{\alpha_i\alpha_j}{p_ip_j}} \geq q^A \quad (21)$$

where $A = -\sum_{i=1}^{m}\sum_{j=1}^{m}(\frac{\alpha_i\alpha_j}{p_i p_j})$

Proofby (18) (H_3) and (19), we have

$$\int_0^q r_i(t)\left|D_q u_i(t)\right|^{p_i} d_q t \le q\int_0^1 f_i(t)\prod_{k=1}^{m}\left|u_k(t)\right|^{\alpha_k} d_q t, i=1,2,...,m$$

It follows from (20) and the H□lder inequality that

$$\left|u_i(t)\right|^{p_i} \le q\frac{\xi_i(\tau)\eta_i(\tau)}{\xi_i(\tau)+\eta_i(\tau)}\int_0^q r_i(t)\left|D_q u_i(t)\right|^{p_i} d_q t$$

$$\int_0^1 f_j^+(t)\left|u_i(t)\right|^{p_2} d_q t \le qM_{ij}\prod_{k=1}^{m}(\int_0^1 f_i^+(t)\left|u_k(t)\right|^{p_k} d_q t)^{\frac{\alpha_k}{p_k}}$$

where

$$M_{ij} = \int_0^1 \frac{\xi_i(t)\eta_i(t)}{\xi_i(t)+\eta_i(t)} f_j^+(t) d_q t, i=1,2,...,m \tag{22}$$

Similar to the proof of the (17), we get

$$\int_0^1 f_i(t)\left|u_k(t)\right|^{p_k} d_q t > 0, i,k=1,2,...,m$$

Therefore

$$\prod_{i=1}^{m}\prod_{j=1}^{m}(qM_{ij})^{\frac{\alpha_i\alpha_j}{p_i p_j}} \ge 1 \tag{23}$$

It follows from (22) and (23) that (21) holds.

Corollary 3.2.Suppose that hypothesis (H_3) are satisfied, if the system(18)has a solution $(u_1(t),u_2(t),...u_m(t))$ satisfies(19), then one has the following inequality:

$$\prod_{i=1}^{m}\prod_{j=1}^{m}(\int_0^1 (\xi_i(t)\eta_i(t))^{\frac{1}{2}} \mathrm{f}_j^+(t) d_q t)^{\frac{\alpha_i\alpha_i}{p_i p_j}} \ge 2q^A$$

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