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A Lyapunov-type inequality for a nonlinear difference equation with $\psi -$ Laplacian operator

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Abstract: We prove a Lyapunov-type inequality for a nonlinear difference equationwith $\psi - Laplacian$ operator where ψ is an odd increasing function which is sub-multiplicative on $[0,\infty)$ and $\Psi(s) = 1/\psi(s)$ is a convex function for $s > 0.$

Keywords: Lyapunov-type inequality, $\psi - Laplacian$ operator, difference equation

1. Introduction

It is a classical topic for us to study Lyapunov-typeinequalities which have proved to be very useful in oscillation theory, disconjugacy, eigenvalue problems and numerous other applications in the theory of differential and difference equations. In recent years, there are many literatures with extended the classicalLyapunov-type inequalities for differential and difference systems. Here, we only mention some references [1-8].

This paper is devoted to establishing Lyapunov-type inequalities for the problem

$$(p_\lambda) \begin{cases} \Delta\psi(\Delta u(t)) + \lambda r(t) f(u(t)) = 0, t \in \mathbf{Z}(a,b), \\ u(a) = u(b) = 0. \end{cases}$$

where $a,b \in \mathbf{Z}, b-a > 2,$ $\psi(s)$ is an odd continuous function which is sub-multiplicative for $s \geq 0,$ the $r(t)$ is positive on $\mathbf{Z}[a,b]$, the nonlinearity $f : \mathbf{R} \to \mathbf{R}$ is an odd continuous function which satisfies a sign condition, and the real number λ is a positive parameter.

We say that u is a solution of (p_λ) if u is defined on $\mathbf{Z}[a,b]$, $\psi(s)$ is absolutely continuous and u satisfies the equation in (p_λ). We will consider the following three assumptions on ψ, f and r.

(H_1) The function $f \in C(\mathbf{R})$ is odd and satisfies $sf(s) > 0$ for $s \neq 0.$

(H_2) The weight $r : \mathbf{Z}[a,b] \to (0,\infty).$

(H_3) The function ψ is odd, increasing, and sub-multiplicative on $[0,\infty)$. (See the definition below) and $\Psi(s) = 1/\psi(s)$ is a convex function for $s > 0.$

The following definition is basic to our approach.

Definition1.1. Let $\mathbf{J}$ be an internalin $\mathbf{R}_+ : [0,\infty)$ such that $\mathbf{J}\cdot\mathbf{J} \subseteq \mathbf{J}.$ A function $g : \mathbf{J} \to \mathbf{R}_+$ is said to be sub-multiplicative on $\mathbf{J}$ if the inequality $g(xy) \le g(x)g(y)$ holds for all $x, y \in \mathbf{J}.$

2. A Lyapunov-type inequality

The main result of the present paper is the following, which generalizes the Lyapunov inequality in the setting of sub-multiplicative functions.

Theorem 2.1. Suppose that conditions (H_1) — (H_3) above are satisfied. If $u(t)$ is a nontrivial solution of problem (p_λ) satisfying $u(t) > 0.$ for $t \in \mathbf{Z}(a,b),$ then the following inequality holds:

$$\frac{2}{\psi(\frac{b-a}{2})} < \lambda \sum_{t=a+1}^{b-1} r(t) \frac{f(u(t))}{\psi(u(t))}.$$

Proof. Since u is a nontrivial solution of $(p_\lambda),$ it follows that $|u(t)|$ attains its maximum on $\mathbf{Z}(a,b),$ i.e. there is a $c \in \mathbf{Z}(a,b)$ such that $|u(c)| = \max_{t\in\mathbf{Z}[a,b]} |u(t)|.$

Since $\Psi(s) = 1/\psi(s)$ is a convex function on $(0,\infty),$ we have that for each two positive numbers *A* and *B*. $\Psi(tA + (1-t)B) \le t\Psi(A) + (1-t)\Psi(B).$ for $0 < t < 1.$

In the last inequality, set $t = \frac{1}{2}, A = c - a$ and $B = b - c.$ By using the fact that $\psi(s)$ is increasing and sub-multiplicative for $s > 0,$ we obtain

$$\frac{2}{\psi(\frac{b-a}{2})} \le \frac{1}{\psi(c-a)} + \frac{1}{\psi(b-c)} = \frac{1}{\psi(u(c))}[\frac{\psi(u(c))}{\psi(c-a)} + \frac{\psi(u(c))}{\psi(b-c)}]$$

$$= \frac{1}{\psi(u(c))}[\frac{\psi((c-a)\frac{u(c)}{(c-a)})}{\psi(c-a)} + \frac{\psi((b-c)\frac{u(c)}{(b-c)})}{\psi(b-c)}] \le \frac{1}{\psi(u(c))}[\psi(\frac{u(c)}{c-a}) + \psi(\frac{u(c)}{b-c})]$$

$$= \frac{1}{\psi(u(c))}[\psi(\frac{u(c)-u(a)}{c-a}) + \psi(\frac{u(c)-u(b)}{b-c})] = \frac{1}{\psi(u(c))}[\psi(\frac{u(c)-u(a)}{c-a}) - \psi(\frac{u(\mathrm{b})-u(\mathrm{c})}{b-c})]$$

$$= \frac{1}{\psi(u(c))}[\psi(\frac{\sum_{\tau=a}^{c-1}\Delta u(\tau)}{c-a}) - \psi(\frac{\sum_{\tau=c}^{b-1}\Delta u(\tau)}{b-c})]$$

Denote $\Delta u(\tau_1)=\max_{t\in\mathbf{Z}[a,c-1]}\Delta u(\tau), \Delta u(\tau_2)=\min_{t\in\mathbf{Z}[c,b-1]}\Delta u(\tau)$

Then

$$\frac{2}{\psi(\frac{b-a}{2})}\le\frac{1}{\psi(u(c))}[\psi(\Delta u(\tau_1))-\psi(\Delta u(\tau_2))]$$

$$=\frac{1}{\psi(u(c))}[-\sum_{\tau=\tau_1}^{c-1}\Delta\psi(\Delta u(\tau))-\sum_{\tau=c}^{\tau_2-1}\Delta\psi(\Delta u(\tau))]=\frac{1}{\psi(u(c))}[-\sum_{\tau=\tau_1}^{\tau_2-1}\Delta\psi(\Delta u(\tau))]$$

$$=\frac{\lambda}{\psi(u(c))}[\sum_{\tau=\tau_1}^{\tau_2-1}r(t)f(u(t))]<\lambda\sum_{t=a+1}^{b-1}\frac{r(t)f(u(t))}{\psi(u(t))}.$$

Remark2.1. Let $u(t)$ be a positive solution of problem (p_λ) and $u(c)=\|u\|_\infty=\max_{t\in Z[a,b]}|u(t)|$. From the proof oftheorem 2.1 we deduce the inequality

$$\psi(\|u\|_\infty)<\frac{\psi(\frac{b-a}{2})}{2}\lambda\sum_{t=a+1}^{b-1}r(t)f(u(t)).$$

Corollary 2.2. let $f_1,f_2:\mathbf{Z}[a,b]\to(0,\infty),\varphi_m(t)=|t|^{m-2}t$ for $m>1$. Let us assume that there exists a positive solution of the system

$$(s_{pq})\begin{cases}-\Delta\varphi_p(\Delta u(t))=f_1(t)|u(t)|^{\alpha-2}u(t)|v(t)|^{\beta}.\\ -\Delta\varphi_q(\Delta v(t))=f_2(t)|u(t)|^{\alpha}|v(t)|^{\beta-2}v(t).\end{cases}$$

on $\mathbf{Z}(a,b)$,with Dirichlet boundary conditions.Then

$$\frac{2^{\alpha+\beta}}{(b-a)^{\frac{\alpha}{p'}+\frac{\beta}{q'}}}\le(\sum_{t=a+1}^{b-1}f_1(t))^{\frac{\alpha}{p}}(\sum_{t=a+1}^{b-1}f_2(t))^{\frac{\beta}{q}}.$$

where the positive parameters α,β satisfy $\frac{\alpha}{p}+\frac{\beta}{q}=1$ and p' denotes theexponent

conjugateto p ,i.e. $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. Let (u,v) be a positive solution of (s_{pq}). Supposing that $r(t)=f_1(t)(v(t))^{\beta}$, $\psi(t)=\varphi_p(t)$ and $f(s)=|s|^{\alpha-2}s$ in problem (p_λ) with $\lambda=1$,we get by theorem 2.1

$$2\varphi_p(\frac{2}{b-a})<\sum_{t=a+1}^{b-1}f_1(t)(v(t))^{\beta}\frac{|u(t)|^{\alpha-2}u(t)}{\varphi_p(u(t))}=\sum_{t=a+1}^{b-1}f_1(t)(v(t))^{\beta}(u(t))^{\alpha-p}\quad(2.1)$$

Similarly, setting $\psi(t)=\varphi_q(t),\ r(t)=f_2(t)(u(t))^{\alpha}$ and $f(s)=|s|^{\beta-2}s$,we obtain the inequality

$$2\varphi_q(\frac{2}{b-a})<\sum_{t=a+1}^{b-1}f_2(t)(u(t))^{\alpha}(v(t))^{\beta-q}\ (2.2)$$

Denote $\max_{t\in Z(a,b)}u(t)=u(d),\ \max_{t\in Z(a,b)}v(t)=v(e)$,

then by (2.1) and (2.2), we have

$$(2\varphi_p(\frac{2}{b-a}))^{\frac{\alpha}{p}}<(\sum_{t=a+1}^{b-1}f_1(t))^{\frac{\alpha}{p}}(v^{\beta}(e)u(d)^{\alpha-p})^{\frac{\alpha}{p}}.\ (2.3)$$

$$(2\varphi_q(\frac{2}{b-a}))^{\frac{\beta}{q}}<(\sum_{t=a+1}^{b-1}f_2(t))^{\frac{\beta}{q}}((v(e))^{\beta-q}u(d)^{\alpha})^{\frac{\beta}{q}}.\ (2.4)$$

since

$$[v^{\beta}(e)u^{\alpha-p}(d)]^{\frac{\alpha}{p}}[v^{\beta-q}(e)u^{\alpha}(d)]^{\frac{\beta}{q}}=v^{\frac{\alpha\beta}{p}}(e)u(d)^{\frac{\alpha^2}{p}-\alpha}v(e)^{\frac{\beta^2}{q}-\beta}u(d)^{\frac{\alpha\beta}{q}}=1.$$

$$2^{\frac{\alpha}{p}}(\frac{2}{b-a})^{(p-1)\frac{\alpha}{p}}2^{\frac{\beta}{q}}(\frac{2}{b-2})^{(q-1)\frac{\beta}{q}}\le(\sum_{t=a+1}^{b-1}f_1(t))^{\frac{\alpha}{p}}(\sum_{t=a+1}^{b-1}f_2(t))^{\frac{\beta}{q}}.$$

i.e. $\dfrac{2^{\alpha+\beta}}{(b-a)^{\frac{\alpha}{p^1}+\frac{\beta}{q^1}}}\le(\sum_{t=a+1}^{b-1}f_1(t))^{\frac{\alpha}{p}}(\sum_{t=a+1}^{b-1}f_2(t))^{\frac{\beta}{q}}.$

3. The number of zeros.

In this section we denote by u_k the restriction of u to a nodal domain I_k of u ,e.g. To a component of $\mathbf{Z}(a,b)\setminus\mathbf{Z}(u),$ where $\mathbf{Z}(u)=\{t\in\mathbf{Z}(a,b);u(t)=0\}$.we obtain bounds for the number of zeros ofnontrivial solution of problem (p_{λ}) with $\lambda=1$.

Corollary3.1. Let $r(t)$ be a positive function on $\mathbf{Z}[a,b]$.Let $u(t)\neq 0$ be a nonnegative solution of problem (p_1). Suppose that u has N zeros on $\mathbf{Z}[a,b]$ with $N>2$. Moreover, suppose that u_k is a solution of problem (p_1) on I_k.

Then

$$N<\{\frac{1}{2}[\sum_{k=1}^{N-1}\psi(\frac{t_{k+1}+t_k}{2})]\sum_{k=1}^{N-1}\sum_{t=t_k+1}^{t_{k+1}-1}r(t)\frac{f(u_k(t))}{\psi(u_k(t))}\}^{\frac{1}{2}}+1.$$

where $a \le t_1 < t_2 < \cdots < t_N \le b$ are the N zeros of u on $\mathbf{Z}[a,b]$ and $I_k = \mathbf{Z}(t_k, t_{k+1})$.

Proof. Applying Theorem 2.1 in each interval I_k ,we have

$$\frac{2}{\psi(\frac{t_{k+1}-t_k}{2})} < \sum_{t=t_k+1}^{t_{k+1}-1} r(t)\frac{f(u_k(t))}{\psi(u_k(t))}. \quad (3.1)$$

for $k = 1,2,\cdots,N-1$. Since the harmonic mean of $N-1$ positive numbers is majorized by their arithmetic mean,we have

$$[\frac{1}{N-1}\sum_{k=1}^{N-1}\frac{1}{\psi(\frac{t_{k+1}-t_k}{2})}]^{-1} \le \frac{1}{N-1}\sum_{k=1}^{N-1}\psi(\frac{t_{k+1}-t_k}{2}).$$

Thus adding (3.1) form $k=1$ to $N-1$ we obtain

$$\frac{2(N-1)^2}{\sum_{k=1}^{N-1}\psi(\frac{t_{k+1}-t_k}{2})} < \sum_{k=1}^{N-1}\sum_{t=t_k+1}^{t_{k+1}-1} r(t)\frac{f(u_k(t))}{\psi(u_k(t))},$$

and hence the result.

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