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CHARACTERIZATION OF BOUNDARY GRAPHS

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Abstract: Let G be a nontrivial connected graph. The distance between two vertices u and v of G is the length of a shortest u-v path in G. Let u be a vertex in G. The eccentricity e(u) of a vertex u is the distance to a vertex farthest from u. A vertex v is an eccentric vertex of u if d(u,v)=e(u), that is every vertex at greatest distance from u is an eccentric vertex of u. A vertex v is an eccentric vertex of G if v is an eccentric vertex of some vertex of G. Consequently,if v is an eccentric vertex of u and w is a neighbor of v, then d(u,w) $\leq$ d(u,v). A vertex v may have this property, however, without being an eccentric vertex of u. A vertex v is a boundary vertex of a vertex u if d(u,w) $\leq$ d(u,v) for all $w \in N(v)$. The boundary graph B(G) based on a connected graph G is a simple graph which has the vertex set as in G. Two vertices u and v are adjacent in B(G) if either u is a boundary of v or v is a boundary of u. If G is disconnected, then each vertex in a component is adjacent to all the vertices in the other components and is adjacent to all of its boundary vertices within the component. A graph G is called a boundary graph if there exists a graph H such that B(H)=G. The main objective of this paper is to solve the equation B(H)=G for a given graph G.

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1. Introduction and Existing Works

The graphs considered here are nontrivial and simple. For other graph theoretical notation and terminology, we follow Buckley [4] and West [17]. In a graph G, the distance d(u,v) between a pair of vertices u and v is the length of a shortest path joining them. The eccentricity e(u) of a vertex u is the distance to a vertex farthest from u. The radius r(G) of G is defined by r(G)= min $\{e(u) : u \in V(G)\}$ and the diameter d(G) of G is defined by d(G)= max $\{e(u) : u \in V(G)\}$.A vertex v is called an eccentric vertex of a vertex u if d(u,v)=e(u). A vertex v of G is called an eccentric vertex of G if it is an eccentric vertex of some vertex of G.

Interconnection networks are pervasive in today's society, including networks for the distribution of goods, communication networks,social networks, and the Internet, to name just a few. The topology of an interconnection network is usually modeled by a graph, either directed or undirected, depending on the particular application. In all cases, there are some common fundamental characteristics of networks such as the number of nodes, number of connections at each node, total number of connections, clustering of nodes etc. Many of the most important basic properties, underpinning the functionalities of a network, are related to the distance between the nodes in a network, such properties includes the eccentricities of the nodes, the radius of the network and the diameter of the network (see [4]).

As a second level of abstraction, binary relations induced by distances in a graph can also represented by a graph. Theoretical research in this direction includes the study of antipodal graphs(see[2,3,17]), antipodal digraphs(see[8]),eccentric graphs (see[1]), eccentric digraphs(see[5,10,11,16]), radial graphs(see[12,13,14]), and radial digraphs(see[15]).

The notion of eccentric graph of a graph G, was introduced by Akiyama et. al.[1]. The eccentric graph of a graph G denoted by G_e, has the same set of vertices as G with two vertices u and v being adjacent in G_e if and only if either v is an eccentric vertex of u or u is an eccentric vertex of v in G; that is d(u,v) = min{e(u),e(v)}. The following results are given in this paper.

1. A few general properties of eccentric graphs.
2. A characterization of graphs G with $G_e = K_p$ and with G_e = pK_2.
3. A solution of the equation $G_e = \overline{G}$.

The concept of antipodal graph was initially introduced by [17] and was further expanded by [2,3]. The antipodal graph of a graph G, denoted by A(G), is the graph on the same vertices as of G, two vertices being adjacent if the distance between them is equal to the diameter of G. If a graph G is disconnected, then the diameter d(G) is ∞. If G is disconnected, then two vertices are adjacent in A(G) if they are in different components of G.

In [8] Garry Johns and Karen sleno introduced the concept of antipodal digraph of a digraph. The definition of antipodal graph is extended to a digraph D where the arc (u,v) is included in A(D) if d(u,v) is the diameter of D. It is shown that a digraph D is an antipodal digraph if and only if D is the antipodal digraph of its complement $\overline{D}$.For any digraph D=(V,E), we define the complement $\overline{D}$ to be the digraph with the same vertex set V, with uv being an arc of $\overline{D}$ if and only if $u \neq v$ and uv is not an arc of D.

For a graph G, its eccentric digraph ED(G) has vertex set V(ED(G)) = V(G) with an arc from v to u if and only if u is an eccentric vertex of v. If u and v are mutually eccentric, then there is a symmetric pair of arcs joining u and v. Fred Buckley [5] presented the eccentric digraphs of many classes of graphs including complete graphs, complete bipartite graphs, antipodal graphs and cycles and gave various interesting general structural properties of eccentric digraphs of graphs. As one of the main result, he mentioned that " For almost every graph G, its eccentric digraph is $ED(G) = \left(\overline{G}\right)^*$, where $\left(\overline{G}\right)^*$ denotes the complement of G in which each undirected edge had been replaced by two symmetric arcs.

James Boland et.al [10] examined eccentric digraphs of igraphs for various families of digraphs and the behaviour of an iterated sequence of eccentric digraphs of a digraph. Joan Gimbert et.al. [11] proved the following theorems.

1. Let G be a graph. Then the eccentric digraph ED(G) is symmetric if and only if G is self-centered.
2. Let G be non-strongly connected digraph. Then ED(G) is symmetric digraph if and only if

$G = C_1 \cup C_2 \cup \ldots \cup C_k\,(k \geq 2)$ or $G = K_n \rightarrow C_1 \cup C_2 \cup \ldots \cup C_k\,(k \geq 1)$ Where $C_{1,} C_2, \ldots C_k$ are strongly connected graphs.

The paper [11] partly characterized graphs with specified maximum degree such that ED(G)=G. As one of the main results, it is ascertained that there exists a self-centered graph G such that ED(G)=G, containing an odd cycle.

Iqbalunnisa et.al [9] introduced the super-eccentric graph of a graph. A graph J(G) is said to be super-eccentric graph of a graph G if V(G) = V(J(G)) and $uv \in E(J(G))$ whenever d(u,v) ≥ r,where r is the radius of the graph G.A graph G is said to be n-eccentric if there are n different eccentricities in that graph. The one eccentric graphs are defined as self centered and always 2-connected. Some of the important results in this paper are

1. For any graph G, J(G) = G_e if and only if G is one of the following:
   (i) G is self-centered.
   (ii) G is bieccentric and for each peripheral point the vertices in the (d-1)th neighborhood must have eccentricity d-1.
2. $J(G) = \overline{G}$ if and only if either G has radius 2 or G is disconnected and each of the component is complete.

Kathiresan and Marimuthu introduced the concept of radial graphs [12]. The radial graph of a graph G denoted by R(G), has the vertex set as in G and two vertices are adjacent in R(G) if the distance between them is equal to the radius of G. If G is disconnected, then two vertices being adjacent in R(G) if they belong to different components. A graph G is called a radial graph if there exists a graph H such that R(H)=G. They gave a necessary and sufficient condition for a graph to be a radial graph. The following results are found in [12].

Theorem A [12] Let G be a graph of order n, then $R(G) = G$ if and only if $G \in F_1$, where F_1 is the set of all graphs of radius 1.

Theorem B [12] Let G be a graph of order n. Then $R(G) = \overline{G}$ if and only if either every vertex of G has eccentricity 2 or G is disconnected in which each component is complete.

Theorem C [12] A graph G is a radial graph if and only if either G is a radial graph of itself or the radial graph of its complement.

Theorem D [12] If $G \in F_{22}$ and $\overline{G} \in F_{23}$, then G is not a radial graph.

Kathiresan and Sumathi [15] defined radial digraphs. For a digraph *D*, the Radial digraph *R*(*D*) of *D* is the digraph with *V* (*R*(*D*)) = *V* (*D*) and $E(R(D)) = \{(u,v)/u,v \in V(D)\ and\ \mathrm{d_D}(u,v) = rad(D)\}$. A digraph D is called a radial digraph if R(H) = D for some digraph H.If there exists a digraph H with finite radius and infinite diameter, such that R(H)=D, then the digraph D is said to be a Radial digraph of type 1. Otherwise D is said to be a Radial digraph of type 2. They proved that if D is a radial digraph of type 2 then D is the radial digraph of itself or the radial digraph of its complement. They generalized the known characterization for radial graphs. Also they characterized self complementary self radial digraphs.

In [14], Kathiresan et al. defined dynamics of radial graphs. Given a positive integer m, the m^{th} iterated radial graph of G is defined as $R^m(G) = R(R^{m-1}(G))$. A graph G is periodic if $R^m(G) \cong G$ for some m. If p is the least positive integer with this property, then G is called a fixed graph. A graph G is said to be an eventually periodic graph if there exists positive integers m and k >0 such that $R^{m+i}(G) \cong R^i(G), \forall i \geq k$. They proved that any graph G is periodic if and only if any of the following holds.

1.G is disconnected with each component complete, and $|V_i| \geq 2,$ for each i^{th} component.

2. $R(G) \cong G$

3. $\overline{G}$ is a connected radial graph with $R(\overline{G}) \cong G.$

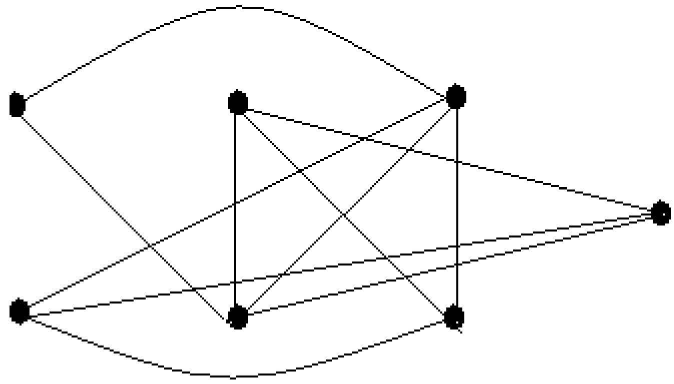
4. G is such that $R(\overline{G}) \cong G$ and $R^m(\overline{G}) \cong R(\overline{G})$ for some $m \geq 2.$

Chartrand et al. [6,7] introduced the concept of boundary vertices in graphs. If v is an eccentric vertex of a vertex u and w is a neighbor of v, then d(u,w) ≤ d(u,v). A vertex v may have this property, however, without being an eccentric vertex of u. A vertex v is a boundary vertex of u if d(u,w) ≤ d(u,v) for all $w \in N(v).$ It is not necessary that the vertices u and v are boundary vertices to each other.

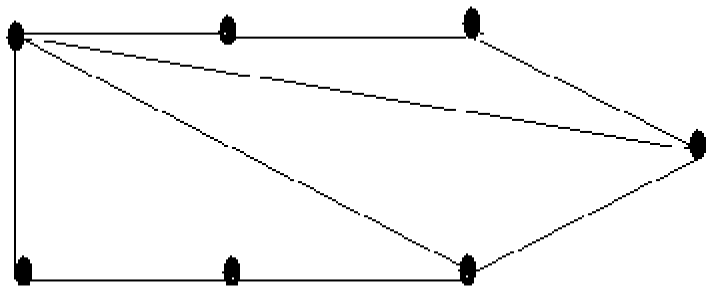
Motivated by the above works, we introduce a new graph called boundary graph B(G) based on a connected graph G.B(G) is a simple graph whose vertex set is V(G) and the two vertices u and v are adjacent in B(G) if either u is a boundary of v or v is a boundary of u. A graph G is called a boundary graph if there exists a graph H such that B(H) = G. If G is disconnected, then each vertex in a component is adjacent to all the vertices in the other components and is adjacent to all of its boundary vertices within the component. In this paper, we denote $G_1 = G_2$ if the two graphs G_1 and G_2 are the same graphs. We define the neighborhood $N_k(u) = \{w \in N(v)/d(u,w) = k\}$.A vertex $v \in V(G)$ is called a complete vertex if $\langle N(v) \rangle$ is complete. It is clear that every complete vertex is a boundary vertex of all other vertices. Note that we do not consider the case $B(G) \cong G.$

Let $F_{11}, F_{12}, F_{22}, F_{23}, F_{24}$ and F_3 denote the set of all graphs G such that r(G) = 1 and d(G) = 1; r(G) = 1 and d(G) = 2; r(G) = 2 and d(G) = 2; r(G) = 2 and d(G) = 3; r(G) = 2 and d(G) = 4; $r(G) \geq 3$ respectively and F_4 denote the set of all disconnected graphs.

Now we show that the notions of radial graph and boundary graph are independent. It is easy to check that for the following graph G r(G) = 2, d(G) = 2, $r(\overline{G}) = 2$ and $d(\overline{G}) = 3.$ By applying Theorem D, G is not a radial graph. But G is the boundary graph of H.



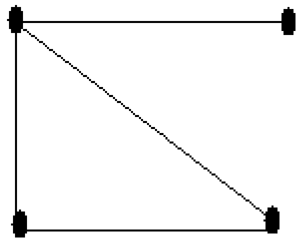
The graph G



The graph H

Figure 1

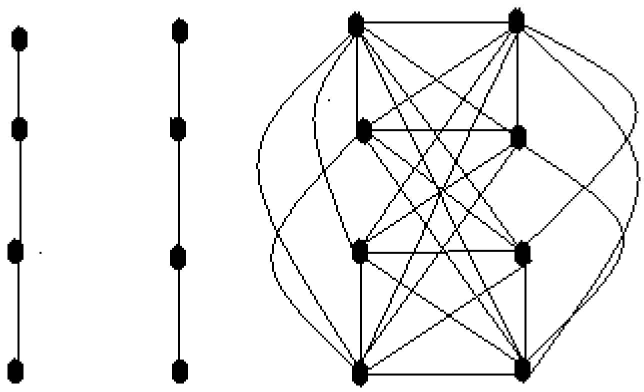
Now we consider the following graph G.



A radial graph but not a boundary graph.

Figure 2

The fact that G is a radial graph is clear from Theorem A. But this is not a boundary graph of any graph on 4 vertices. Now we give an example of a disconnected graph and its boundary graph B(G).



The graph G The graph B(G)

Figure 3

2. Boundary graph of some classes of graphs

This section gives the boundary graph of some classes of graphs.

Result 2.1. B(G) = G if and only if G is complete.

Proof. Suppose that G is not complete.Then $G \in F_{12} \cup F_{22} \cup F_{23} \cup F_{24} \cup F_3$. If $G \in F_{12}$, then either $B(G) = K_n$ *or* $B(G) \in F_{12}$.Our assumption shows that $B(G) \in F_{12}$. But there is at least one pair of non-adjacent vertices of eccentricity 2 in G. These two vertices are adjacent in B(G). Therefore $B(G) \neq G.$ If

$G \in F_{22}$ with at least one complete vertex, then $B(G) \in F_{12}$. If $G \in F_{22}$ has no complete vertices, then there exist at least one pair of non-adjacent vertices u and v of eccentricity 2. This shows that $uv \in B(G)$. Thus in this case, we obtain $B(G) \neq G$. Similar argument can be applied if $G \in F_{23} \cup F_{24} \cup F_3$. Conversely, assume that G is Complete.For a vertex u in a complete graph G, every vertex v other than u is a boundary vertex of u and hence B(G) = G.

There are non-complete graphs G of order n for which $B(G) = K_n$.

Result 2.2. $B(K_{1,n}) = K_{n+1}$ for every n. □

Result 2.3. $B(C_n) = \cup \frac{n}{2} K_2$ if n is even

$B(C_n) \cong C_n$, if n is odd. □

By observing the Results 2.1 and 2.3, we have the following problem.

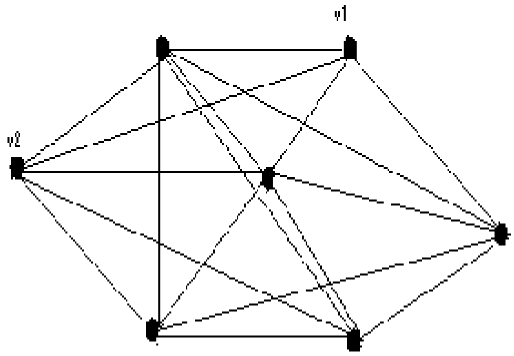
Problem1. Characterize all graphs G for which $B(G) \cong G$.

Result 2.4. $B(K_{m,n}) = K_m \cup K_n$, $m, n \geq 2.$ □

The proof of following proposition is obvious.

Proposition 2.5. Every vertex of G is complete if and only if G is complete.

For the graph G given in Figure 4, $r(G) = 1, d(G) = 2,$ $e(v_1) = e(v_2) = 2,$ $B(G) \neq K_n$, since v_1 *and* v_2 are non-adjacent in B(G).



A graph $G \in F_{12}$ for which B(G) is non complete.

Figure 4

Theorem 2.6. For a graph $G \in F_{12}$, $B(G) = K_n$ if and only if either $N(u) - \{v\} \subseteq N(v) - \{u\}$ or $N(v) - \{u\} \subseteq N(u) - \{v\}$ for any two adjacent vertices u and v of G.

Proof. Suppose that for any two adjacent vertices u and v of G either $N(u) - \{v\} \subseteq N(v) - \{u\}$ or $N(v) - \{u\} \subseteq N(u) - \{v\}$.If $N(u) - \{v\} \subseteq N(v) - \{u\}$.Then d(v,u) = d(w,v) for all $w \in N(u)$. This implies that u is a boundary of v. Consider $N(v) - \{u\} \subseteq N(u) - \{v\}$.In the same way , we can prove v is a boundary of u. Therefore $uv \in B(G)$. This shows that if $uv \in E(G)$, then $uv \in E(B(G))$. Also any two non-adjacent vertices are boundary to each other in G. Hence they are adjacent in B(G).

Suppose that for any two adjacent vertices u and v of G neither $N(u)-\{v\}\subseteq N(v)-\{u\}$ nor $N(v)-\{u\}\subseteq N(u)-\{v\}$. Then d(v,u) < d(v,w) for some $w\in N(u)$ and d(u,v) < d(u,w) for some $w\in N(v)$. Therefore $uv\notin B(G)$, which is a contradiction to the fact that $B(G)=K_n$.

□

3. A necessary and sufficient condition for a graph to be a boundary graph

This section provides a tool that check whether a given graph is a boundary graph or not.Next we give a graph $G\in F_{12}$ with no complete vertex and its boundary graph B(G). Note that B(G) is not equal to the complement of G.

The graph $G\in F_{12}$ The graph $B(G)\neq\overline{G}$

Figure 5

Lemma 3.1. Let G be a graph. Then $B(G)=\overline{G}$ if and only if the following conditions hold.

(i) G has no complete vertex.

(ii) neither $N(u)-\{v\}\subseteq N(v)-\{u\}$ nor $N(v)-\{u\}\subseteq N(u)-\{v\}$ for any two adjacent vertices u and v of G.

(iii) either $N_k(u)=\phi$ or $N_k(v)=\phi$ for any two non-adjacent vertices u and v of G, where k= d(u,v)+1.

Proof. Assume that $B(G)=\overline{G}$. Suppose (i) does not hold. Then $B(G)\in F_{12}$ which is a contradiction.Suppose (ii) does not hold. Then either $N(u)-\{v\}\subseteq N(v)-\{u\}$ or $N(v)-\{u\}\subseteq N(u)-\{v\}$ for some adjacent vertices u and v of G. If $N(u)-\{v\}\subseteq N(v)-\{u\}$, then d(v,u) = d(w,v) for all $w\in N(u)$. This shows that u is a boundary of v. In the same way we can prove v is a boundary of u if $N(v)-\{u\}\subseteq N(u)-\{v\}$. Therefore $uv\in B(G)$ which is a contradiction to the fact that $B(G)=\overline{G}$.

Suppose (iii) does not hold. Then neither $N_k(u)=\phi$ nor $N_k(v)=\phi$ for some non-adjacent vertices u and v of G, where k= d(u,v)+1. If $N_k(u)\neq\phi$, then there exists at least one element $w\in N_k(u)$ such that d(u,w) = k. This implies $d(u,v)<d(u,w)$. Hence v is not a boundary of u. Similarly u is not a boundary of v if $N_k(v)\neq\phi$. Therefore $uv\notin B(G)$, which is a contradiction to the fact that $B(G)=\overline{G}$. The proof of the converse part is obvious. □

Lemma 3.2. Every graph $G\in F_{11}$ is a boundary graph.

Proof. The proof follows from Result 2.1. □

Lemma 3.3. If $H\in F_4$, then B(H) is a connected graph.

Proof. Follows from the definition. □

Lemma 3.4. If G has at least one isolated vertex, then G is not a boundary graph.

Proof. Let G be a graph with at least one isolated vertex. Suppose that G is a boundary graph. Then there exists a graph H such that B(H) = G. By Lemma 3.3. B(H) is connected which is a contradiction. Therefore G is not a boundary graph.

Lemma 3.5. Let $G \in F_4$ without isolated vertices. If $\overline{G}$ has the following properties

(i) $\overline{G}$ has no complete vertex.

(ii) neither $N(u)-\{v\} \subseteq N(v)-\{u\}$ nor $N(v)-\{u\} \subseteq N(u)-\{v\}$ for any two adjacent vertices u and v of $\overline{G}$.

(iii) either $N_k(u)=\phi$ or $N_k(v)=\phi$ for any two non-adjacent vertices u and v of $\overline{G}$, where k= d (u,v)+1, then G is a boundary graph.

Proof. By Lemma 3.1, $B(\overline{G}) = G$. □

Theorem 3.6. A graph G is a boundary graph if and only if it is the boundary graph of its complement $\overline{G}$ with the following properties

(i) $\overline{G}$ has no complete vertex.

(ii) neither $N(u)-\{v\} \subseteq N(v)-\{u\}$ nor $N(v)-\{u\} \subseteq N(u)-\{v\}$ for any two adjacent vertices u and v of $\overline{G}$.

(iii) either $N_k(u)=\phi$ or $N_k(v)=\phi$ for any two non-adjacent vertices u and v of $\overline{G}$, where k= d (u,v)+1.

Proof. Suppose there exists a graph H such that B(H) = G.We have to prove that G is the boundary graph of its complement $\overline{G}$ with the properties (i), (ii) and (iii). Suppose not. Then either $H \neq \overline{G}$ and H has the properties (i),(ii) and (iii) or $H = \overline{G}$ and one of the property fails.

Case (i) If $H \neq \overline{G}$ and H has the properties (i),(ii) and (iii), then by Lemma 3.1 $B(H) = \overline{H}$. This implies $\overline{H} = G$. Therefore $H = \overline{G}$ which is a contradiction.

Case (ii) If $H = \overline{G}$ and any one of the property fails. Suppose (i) does not hold for $\overline{G}$. Let u be a complete vertex in $\overline{G}$. Then u is adjacent with all the vertices in $B(\overline{G})$. This implies $B(\overline{G}) \neq G$ which is a contradiction. Suppose (ii) does not hold for $\overline{G}$, then either $N(u)-\{v\} \subseteq N(v)-\{u\}$ or $N(v)-\{u\} \subseteq N(u)-\{v\}$ for some adjacent vertices u and v of $\overline{G}$. This implies that u and v are adjacent in $B(\overline{G})$. Hence $B(\overline{G}) \neq G$ which is a contradiction.Suppose (iii) does not hold for $\overline{G}$, then neither $N_k(u)=\phi$ nor $N_k(v)=\phi$ for some non-adjacent vertices u and v of $\overline{G}$, where k= d (u,v)+1. Then $uv \notin \overline{G}$ and $uv \notin B(\overline{G})$. This shows that $B(\overline{G}) \neq G$. Hence G is not a boundary graph which is a contradiction.

The proof of the converse part follows from Lemmata 3.1, 3.4, and 3.5.

4.Conclusion

In this paper we have solved the graph equation B(H) = G for a given graph G. This paper contains an interesting open problem that characterize all graphs G for which $B(G) \cong G$. One can investigate the properties of iterations of boundary graphs. For detailed description of dynamics of graph operators, one can refer [19].

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